Consideration on singularities in learning theory and real log canonical threshold

Miki Aoyagi

Nihon University

log canonical threshold

Definition

Y : a smooth variety

f : a nonzero regular fucntion on Y.

$Z \subset Y$: a closed subscheme

The log canonical threshold $c_Z(Y, f)$

$$c_Z(Y_{\mathbb{C}}, f) = \sup\{c : |f|^{-c} \text{ is locally } L^2 \text{ on n.b.d. of } Z\}.$$

$$c_Z(Y_{\mathbb{R}}, f) = \sup\{c : |f|^{-c} \text{ is locally } L^1 \text{ on n.b.d. of } Z\}.$$

$(\int |z^m|^{-2c} dz d\overline{z} < +\infty, \int |x^m|^{-c} dx < +\infty \leftrightarrow c < 1/m)$

$\pi : Y' \to Y$

$K_{Y'} = \pi^*(K_Y) + \sum a_i E_i, \qquad \pi^*(f = 0) = \sum b_i E_i$ normal crossing,

$c_Z(Y, f) = \min_{\pi(E_i) \cap Z \neq \varnothing} \{\frac{a_i + 1}{b_i}\} \qquad 0 < c_Z(Y_{\mathbb{C}}, f) \leq 1, 0 < c_Z(Y_{\mathbb{R}}, f)$

Bernstein-Sato polynomial

Theorem (Bernstein 71, Björk79)

$$I := \langle b(s) \in \mathbb{C}[s] : b(s) f^s = P f^{s+1},$$
$$P : \text{a linear differential operator} \rangle.$$

Bernstein-Sato polynomial $b_f(s)$:

$I = \langle b_f(s) \rangle$ with leading coefficient 1

Theorem (Kashiwara76,Lichtin89)

$$c_0(\mathbb{C}^d, f) = \text{the largest root of } b_f(s)$$
$$(c_0(\mathbb{R}^d, f) = \text{the root of } b_f(s) \bmod 1 \ )$$

Theorem Atiyah, Bernstein, Sato & Shintani, Björk

Theorem Atiyah, Bernstein, Sato & Shintani, Björk

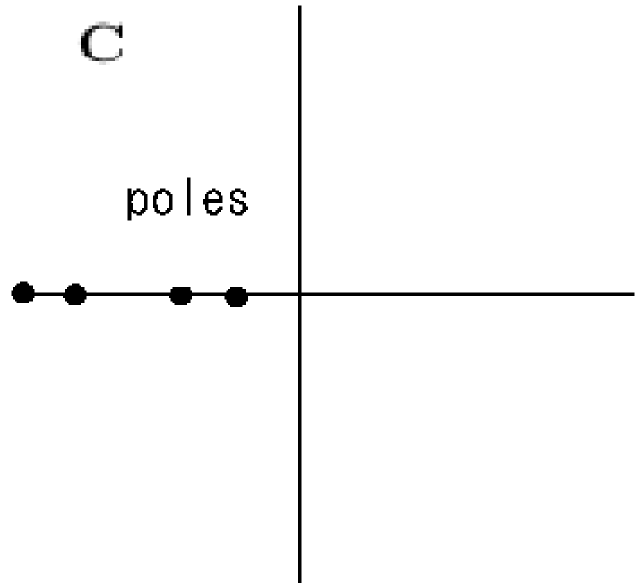
$\psi : C^{\infty}-$ function with compact support.

Theorem Atiyah, Bernstein, Sato & Shintani, Björk

$\psi : C^\infty -$ function with compact support.

$$\zeta(z) = \int_{\text{n.b.d. of } z} |f_{\mathbb{C}}(w)|^z \psi dw \wedge d\overline{w}$$
$$\zeta(z) = \int_{\text{n.b.d. of } z} |f_{\mathbb{R}}(w)|^z \psi dw$$

are holomorphic in the right-half plane.

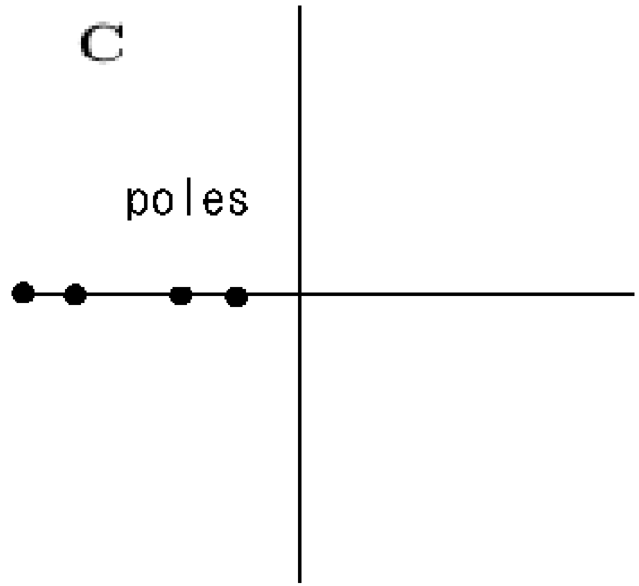


Theorem Atiyah, Bernstein, Sato & Shintani, Björk

$\psi : C^\infty-$ function with compact support.

$$\zeta(z) = \int_{\text{n.b.d. of } Z} |f_{\mathbb{C}}(w)|^z \psi dw \wedge d\overline{w}$$
$$\zeta(z) = \int_{\text{n.b.d. of } Z} |f_{\mathbb{R}}(w)|^z \psi dw$$

are holomorphic in the right-half plane.



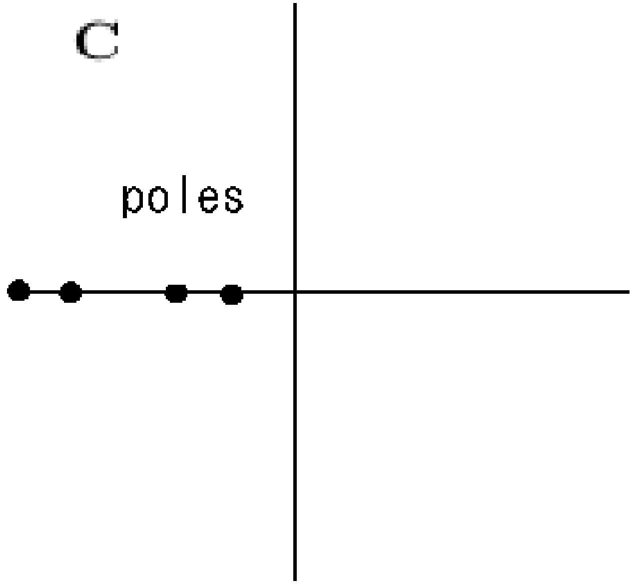
$\zeta(z)$ is a meromorphic function on $\mathbf{C}$.
Its poles are negative rational numbers.

Theorem Atiyah, Bernstein, Sato & Shintani, Björk

$\psi : C^\infty -$ function with compact support.

$$\zeta(z) = \int_{\text{n.b.d. of } Z} |f_{\mathbb{C}}(w)|^z \psi dw \wedge d\overline{w}$$
$$\zeta(z) = \int_{\text{n.b.d. of } Z} |f_{\mathbb{R}}(w)|^z \psi dw$$

are holomorphic in the right-half plane.



$\zeta(z)$ is a meromorphic function on $\mathbf{C}$.
Its poles are negative rational numbers.

$-c_Z(\mathbb{C}^n, f) =$ the largest pole of $\int_{\text{n.b.d. of } Z} |f|^{2z} \psi dw d\overline{w}$
$-c_Z(\mathbb{R}^n, f) =$ the largest pole of $\int_{\text{n.b.d. of } Z} |f|^{z} \psi dw$

Hironaka Theorem

Hironaka Theorem

f : analytic in a neighborhood of $w \in \mathbf{R}^d$ with $f(w) = 0$.

Hironaka Theorem

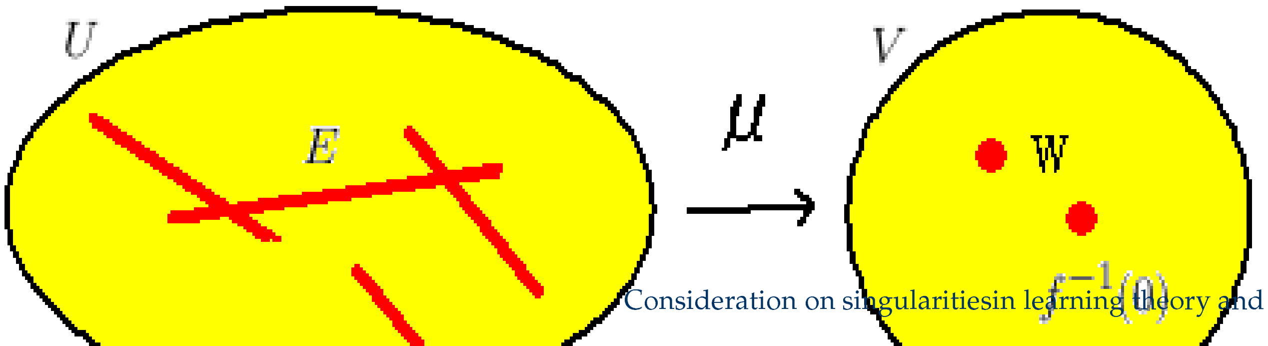
f : analytic in a neighborhood of $w \in \mathbf{R}^d$ with $f(w) = 0$.

$$\text{Then there exist } \begin{cases} \text{an open set } V \ni w, \\ \text{a analytic manifold } U, \\ \text{a proper analytic map } \mu \end{cases}$$

such that

$$\mu : U - E \xrightarrow{\sim} V - f^{-1}(0), \qquad E = \mu^{-1}(f^{-1}(0))$$

$$f(\mu(u_1, u_2, \cdots, u_d)) = \pm u_1^{s_1} u_2^{s_2} \cdots u_n^{s_d}, s_i \in \mathbf{Z}_{\geq 0}.$$



Apply Hironaka's Theorem

$$\zeta(z) = \int_V |f(w)|^z \psi(w) dw$$

$$= \sum_{U_u} \int_{U_u} |u_1^{s_1} u_2^{s_2} \cdots u_d^{s_d}|^z u_1^{k_1} u_2^{k_2} \cdots u_d^{k_d} du.$$

We have poles $-\frac{k_1+1}{s_1}, -\frac{k_2+1}{s_2}, \ldots, -\frac{k_d+1}{s_d}$

Fact

- $I = \langle f_1, \ldots, f_m \rangle$
  $c_Z(Y_{\mathbf{R}}, I) := c_Z(Y, f_1^2 + \cdots + f_m^2)$
- A : a matrix, P, Q : regular matrices
  $c_Z(Y_{\mathbf{R}}, ||A||^2) = c_Z(Y, ||PAQ||^2)$

Definition (Vandermonde matrix type singularities)

a^*, b^* : constants

Fix $Q \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1H} & a^*_{1,H+1} & \cdots & a^*_{1,H+r} \\ a_{21} & \cdots & a_{2H} & a^*_{2,H+1} & \cdots & a^*_{2,H+r} \\ & \vdots & & & \vdots & \\ a_{M1} & \cdots & a_{MH} & a^*_{M,H+1} & \cdots & a^*_{M,H+r} \end{pmatrix},$$

$I = (\ell_1, \ldots, \ell_N) \in (\mathbb{N} \cup \{0\})^N$,

$B_I =$
$(\prod_{j=1}^N b_{1j}^{\ell_j}, \prod_{j=1}^N b_{2j}^{\ell_j}, \cdots, \prod_{j=1}^N b_{Hj}^{\ell_j}, \prod_{j=1}^N {b^*_{H+1,j}}^{\ell_j}, \cdots, \prod_{j=1}^N {b^*_{H+r,j}}^{\ell_j})^t$ and

$B = (B_I)_{\ell_1+\cdots+\ell_N = Qn+m, 0 \le n \le H+r}$

We call $||AB||^2 = 0$ Vandermonde matrix type singularities

Example

$\langle C \rangle$: ideal generated by $\{c_{ij}\}$.

Remark $\langle AB \rangle = \langle AB' \rangle$ where

$B' = (B_I)_{\ell_1+\cdots+\ell_N=Qn+m, 0\le n \le H'}$ and $H' \geq H + r$.

Example

$N = 1$, $m = 0$, $Q = 1$ and $r = 0$,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1H} \\ & \vdots & \\ a_{M1} & \cdots & a_{MH} \end{pmatrix}, B = \begin{pmatrix} 1 & b_{11} & b_{11}^2 & \cdots & b_{11}^{H-1} \\ 1 & b_{21} & b_{21}^2 & \cdots & b_{21}^{H-1} \\ & & & \vdots & \\ 1 & b_{H1} & b_{H1}^2 & \cdots & b_{H1}^{H-1} \end{pmatrix}$$

Example

$N = 2$, $Q = 2$ and $m = r = H = 1$,

$$A = \begin{pmatrix} a_{11} & a^*_{12} \\ a_{21} & a^*_{22} \\ \vdots & \vdots \\ a_{M1} & a^*_{M,2} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b^3_{11} & b_{12} & b^3_{12} & b_{11}b^2_{12} & b^2_{11}b_{12} \\ b^*_{21} & {b^*_{21}}^3 & b^*_{22} & {b^*_{22}}^3 & b^*_{21}{b^*_{22}}^2 & {b^*_{21}}^2 b^*_{22} \end{pmatrix}.$$

Example

$N = 3, m = Q = r = 1, H = 2$

$$A = \begin{pmatrix} a_{11} & a_{12} & a^*_{13} \\ a_{21} & a_{12} & a^*_{23} \\ \vdots & \vdots & \\ a_{M1} & a_{12} & a^*_{M3} \end{pmatrix}$$

$B =$

$$\begin{pmatrix} b_{11} & b_{11}^2 & b_{12} & b_{12}^2 & b_{13} & b_{13}^2 & b_{11}b_{12} & b_{11}b_{13} & b_{12}b_{13} & b_{11}^3 & \cdots \\ b_{21} & {b_{21}}^2 & b_{22} & {b_{22}}^2 & b_{23} & {b^*_{23}}^2 & b_{21}b_{22} & b_{11}b_{23} & b_{22}b_{23} & b_{21}^3 & \cdots \\ b^*_{31} & {b^*_{31}}^2 & b^*_{32} & {b^*_{32}}^2 & b^*_{33} & {b^*_{33}}^2 & b^*_{31}b^*_{32} & b^*_{11}b^*_{33} & b^*_{32}b^*_{33} & {b^*_{31}}^3 & \cdots \end{pmatrix}$$

Remark Their singularities are not isolated. Degenerate with respect to their Newton polyhedrons

Three layered neural network

Three layered neural network :

N input, H hidden and M output

$$p(y|x,w) = \frac{1}{(2\pi)^{M/2}} \exp(-\frac{1}{2}\sum_{k=1}^{M}(y_k - \sum_{i=1}^{H} a_{ki} \tanh(\sum_{j=1}^{N} b_{ij}x_j))^2).$$

True distribution with r hidden units:

$$p(y|x,w^*) = \frac{1}{(2\pi)^{M/2}} \exp(-\frac{1}{2}\sum_{k=1}^{M}(y_k - \sum_{i=1}^{r} a^*_{k,H+i} \tanh(\sum_{j=1}^{N} b^*_{H+i,j}x_j))^2$$

$\boxed{Q = 2, m = 1.}$ $(\tanh(x) = c_1 x + c_3 x^3 + \cdots)$

Normal mixture model

Normal mixture model :

$$p(x|w) = \frac{1}{(2\pi)^{N/2}} \sum_{i=1}^{H} a_{1i} \exp(-\frac{\sum_{j=1}^{N}(x_j - b_{ij})^2}{2}),$$

$\sum_{i=1}^{H} a_{1i} = 1.$

True distribution with r peaks:

$$p(x|w^*) = \frac{1}{(2\pi)^{N/2}} \sum_{i=H+1}^{H+r} a^*_{1i} \exp(-\frac{\sum_{j=1}^{N}(x_j - b^*_{ij})^2}{2}),$$

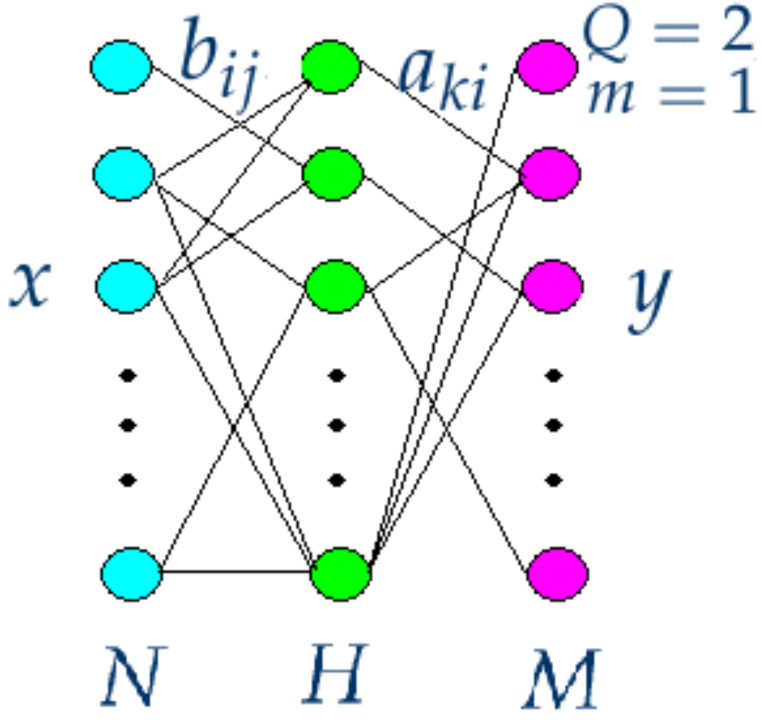
$\sum_{i=H+1}^{H+r} a^*_{1i} = 1.$

$Q = 1, m = 1$ and $M = 1.$

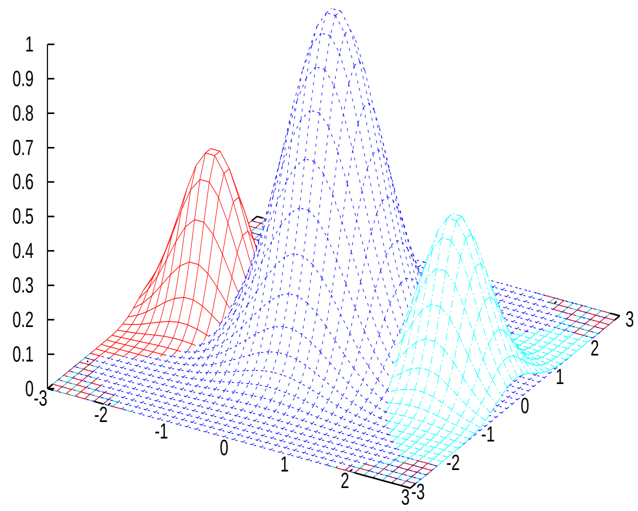
Learning model

$Q = 1, m = 1, M = 1,$
$N =$dim. $H =$ peak's number

Neural Network



Normal mixture model



Theorem 1

Consider a neighborhood of $w^* = \{a^*_{ki}, b^*_{ij}\}$. Assume

$$\left.\begin{array}{c}(b^*_{11}, \cdots, b^*_{1N}) \\ \vdots \\ (b^*_{H_0 1}, \cdots, b^*_{H_0 N})\end{array}\right\} = 0, \quad \left.\begin{array}{c}(b^*_{H_0+1,1}, \cdots, b^*_{H_0+1,N}) \\ \vdots \\ (b^*_{H_0+H_1,1}, \cdots, b^*_{H_0+H_1,N})\end{array}\right\} = w^*_1,$$

$$\cdots, \left.\begin{array}{c}(b^*_{H_0+\cdots+H_{r'-1}+1,1}, \cdots, b^*_{H_0+\cdots+H_{r'-1}+1,N}) \\ \vdots \\ (b^*_{H_0+\cdots+H_{r'-1}+H_{r'},1}, \cdots, b^*_{H_0+\cdots+H_{r'-1}+H_{r'},N})\end{array}\right\} = w^*_{r'}.$$

and $H_0 + \cdots + H_{r'} = H$.

Then $c_{w^*}(||AB||^2) = \sum_{i=1}^{r'} c_{w^*_i}(||A_i B_i||^2)$

A_i B_i correspond to w^*_i

Theorem 2

$$\text{bound}_{\mathbf{i}} = \begin{cases} \dfrac{MH}{2}, & \text{if } mM \leq N-1, \text{ and } \mathbf{i}=1, \\ \dfrac{mM(H-1)+N}{2m}, & \text{if } mM \leq N-1, \text{ and } \mathbf{i}=2, \\ \dfrac{NH}{2m}, & \text{if } N \leq mM \leq m(N-1), \\ \dfrac{NH}{2m}, & \text{if } M \geq N, (N-1)(m-1) \geq 1, \\ \dfrac{2HN+Q(M(1+k)+(N-1)(2H-k-1))k}{4Qk+4m}, & \\ & \text{if } M \geq N, (N-1)(m-1) = 0, \end{cases}$$

for $\mathbf{i} = 1, 2$, where $k = \max\{i \in \mathbb{Z}; 2H \geq (Qi(i-1)+2mi)(M-N+1)\}$.

$$\text{bound}_3 = \frac{NH + \sum_{i=0}^{k'-1} MQ(k'-i)\binom{N+m+Qi-1}{N-1}}{2m+2Qk'},$$

where $k' = \max\{i \in \mathbb{Z}; NH \geq M\sum_{i'=0}^{i-1}(m+Qi')\binom{N+m+Qi'-1}{N-1}\}$.

Theorem 2

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1H} \\ & \vdots & \\ a_{M1} & \cdots & a_{MH} \end{pmatrix} \quad \begin{array}{l} B_I = (\prod_{j=1}^N b_{1j}^{\ell_j}, \prod_{j=1}^N b_{2j}^{\ell_j}, \cdots, \prod_{j=1}^N b_{Hj}^{\ell_j})^t, \\ B = (B_I)_{\ell_1+\cdots+\ell_N = Qn+m, 0 \le n \le H+r} \end{array}$$

$$c_0(||AB||^2) \le \min\{\text{bound}_1, \text{bound}_3\}.$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1H-1} & a^*_{1,H} \\ & \vdots & & \\ a_{M1} & \cdots & a_{MH-1} & a^*_{M,H} \end{pmatrix} \begin{array}{l} B_I = (\prod_{j=1}^N b_{1j}^{\ell_j}, \prod_{j=1}^N b_{2j}^{\ell_j}, \cdots, \prod_{j=1}^N b_{Hj}^{\ell_j}) \\ , \\ B = (B_I)_{\ell_1+\cdots+\ell_N = Qn+m, 0 \le n \le H+r} \end{array}$$

$$c_0(||AB||^2) \le \min\{\text{bound}_2, \text{bound}_3\}.$$

Theorem 3: Exact Values 1

If $N = 1$, we have

$$c_{w^*}(||AB||^2) = \frac{MQk_0(k_0+1)+2H_0}{4(m+k_0Q)} + \frac{Mr'}{2}$$

$$+\sum_{\alpha=1}^{r}\frac{Mk_\alpha(k_\alpha+1)+2H_\alpha}{4(1+k_\alpha)} + \sum_{\alpha=r+1}^{r'}\frac{Mk_\alpha(k_\alpha+1)+2(H_\alpha-1)}{4(1+k_\alpha)}.$$

$$k_0 = \max\{i \in \mathbb{Z}; 2H_0 \geq M(i(i-1)Q+2mi)\},$$

$$k_\alpha = \max\{i \in \mathbb{Z}; 2H_\alpha \geq M(i^2+i)\},$$

$$k_\alpha = \max\{i \in \mathbb{Z}; 2(H_\alpha - 1) \geq M(i^2+i)\}.$$

Theorem 3: Exact Values 2

Case 1 $H=1$.

$$\lambda(\|A_{M1}B_{1N}\|^2)=\min\{\frac{M}{2},\frac{N}{2m}\},\theta=\begin{cases}1, & \text{if } mM\neq N,\\ 2, & \text{if } mM=N,\end{cases}$$

$$\lambda(\|\mathbf{a}^*B_{1N}\|^2)=\frac{N}{2m},\theta=1.$$

Case 2 $H=2$. $\lambda=c_0(\|A_{M2}B_{2N}\|^2)$.

1. If $mM\leq N-1$ then $\lambda=M$ and $\theta=1$.
2. If $m=1$, $M=N$, then $\lambda=\frac{2N+Q(2N-1)}{2(Q+1)}$ and $\theta=1$.
3. If $m=1$, $N=M-1$ then $\lambda=N$ and $\theta=2$.
4. If $m=1$, $N<M-1$ then $\lambda=N$ and $\theta=1$.
5. If $m=2$, $N=1$, $M=1$, then $\lambda=\frac{1}{2}$ and $\theta=2$.
6. If $m=2$, $N<mM$, $M>1$ then $\lambda=\frac{N}{m}$ and $\theta=1$.
7. If $m\geq 2$, $N=mM$ then $\lambda=\frac{N}{m}$ and $\theta=3$.
8. If $m>2$, $N<mM$ then $\lambda=\frac{N}{m}$ and $\theta=1$.

Theorem 3: Exact Values 3

$\lambda = c_0(\|(A_{M1}, \mathbf{a}^*)B_{2N}\|^2)$,

1. If $m \geq 2, mM \leq N-1$ then $\lambda = \frac{mM+N}{2m}$ and $\theta = 1$.
2. If $m = 1, N \geq M+Q+1$ then $\lambda = \frac{N+M}{2}$ and $\theta = 1$.
3. If $m = 1, N = M+Q$ then $\lambda = \frac{N+M}{2}$ and $\theta = 2$.
4. If $m = 1, M+1 \leq N \leq M+Q-1$ then $\lambda = \frac{2N+Q(2N-1)}{2(Q+1)}$ and $\theta = 1$.
5. If $m = 1, N = M$ then $\lambda = \frac{2N+Q(2N-1)}{2(Q+1)}$ and $\theta = 1$.
6. If $m = 1, N = M-1$ then $\lambda = N$ and $\theta = 2$.
7. If $m = 1, N < M-1$ then $\lambda = N$ and $\theta = 1$.
8. If $m = 2, N = 1, M = 1$, then $\lambda = \frac{1}{2}$ and $\theta = 2$.
9. If $m = 2, N < mM, M > 1$ then $\lambda = \frac{N}{m}$ and $\theta = 1$.
10. If $m \geq 2, N = mM$ then $\lambda = \frac{N}{m}$ and $\theta = 2$.
11. If $m > 2, N < mM$ then $\lambda = \frac{N}{m}$ and $\theta = 1$.

Three layered Neural Network

True distribution : r hidden units

$$\text{bound}_0 = \frac{Mr}{2} + \frac{H(N-1)}{2} + \min\{\frac{r}{2} + \frac{H-r+(M-N+1)(k'+k'^2)}{4k'+2}, \frac{r-1}{2} + \frac{2(H-r+1)+(M-N+1)(k''+k''^2)}{4k''+4}\},$$

where $k' = \max\{i \in \mathbb{Z}; H-r \geq i^2(M-N+1)\}$ and $k'' = \max\{i \in \mathbb{Z}; 2(H-r-1) \geq (i^2+i)(M-N+1)\}$.

Also let $\text{bound}_3 = \dfrac{N(H-r) + \sum_{i=0}^{k'-1} 2M(k'-i)\left\langle {N+2i \atop N-1} \right\rangle}{2+4k'}$,

where $k' = \max\{i \in \mathbb{Z}; N(H-r) \geq M\sum_{i'=0}^{i-1}(1+2i')\left\langle {N+2i' \atop N-1} \right\rangle\}$.

If $M < N$, then $\lambda \leq \min\{\frac{MH+Nr}{2}, \frac{(M+N)r}{2} + \text{bound}_3\}$.

If $M \geq N$, then $\lambda \leq \min\{\text{bound}_0, \frac{(M+N)r}{2} + \text{bound}_3\}$.

Three Layered Neural Network

1. $H-r=0: \lambda = r(\frac{M+N}{2}), \theta = 1.$
2. $H=1, r=0: \lambda = \min\{\frac{M}{2}, \frac{N}{2}\}, \theta = \begin{cases} 1, & \text{if } M \neq N, \\ 2, & \text{if } M = N. \end{cases}$
3. $H-r=1, r \geq 1:$
   (a) If $N > M+1$ then $\lambda = (r-1)(\frac{M+N}{2}) + \frac{2M+N}{2}$ and $\theta = 1$.
   (b) If $N = M+1$ then $\lambda = (r-1)(\frac{M+N}{2}) + \frac{2M+N}{2}$ and $\theta = 2$.
   (c) If $N = M$ then $\lambda = (r-1)(\frac{M+N}{2}) + \frac{3M+3N-1}{4}$ and $\theta = 1$.
   (d) If $N = M-1$ then $\lambda = (r-1)(\frac{M+N}{2}) + \frac{M+2N}{2}$ and $\theta = 2$.
   (e) If $N < M-1$ then $\lambda = (r-1)(\frac{M+N}{2}) + \frac{M+2N}{2}$ and $\theta = 1$.
4. $H=2, r=0:$
   (a) If $N \geq M+1$ then $\lambda = M$ and $\theta = 1$.
   (b) If $N = M$ then $\lambda = \frac{3M-1}{3}$ and $\theta = 1$.
   (c) If $N = M-1$ then $\lambda = N$ and $\theta = 2$.
   (d) If $N < M-1$ then $\lambda = N$ and $\theta = 1$.

Normal Mixture Model

$$\text{bound}_3 = \frac{N(H-r+1) + \sum_{i=0}^{k'-1}(k'-i)\left\langle {N+i \atop N-1} \right\rangle}{2+2k'},$$

where $k' = \max\{i \in \mathbb{Z}; N(H-r+1) \geq \sum_{i'=0}^{i-1}(1+i')\left\langle {N+i' \atop N-1} \right\rangle\}$.

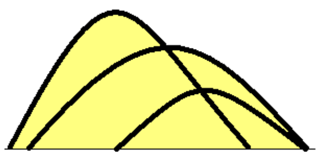
If $1 < N$, then $\lambda \leq \min\{\frac{H-1+Nr}{2}, \frac{(N+1)(r-1)}{2} + \text{bound}_3\}$.

If $N=1$, then

$$\lambda = r-1+\frac{i+i^2+2(H-(r-1))}{4(i+1)}, \theta = \begin{cases} 1, & \text{if } i^2+i < 2(H-(r-1 \\ 2, & \text{if } i^2+i = 2(H-(r-1 \end{cases}$$

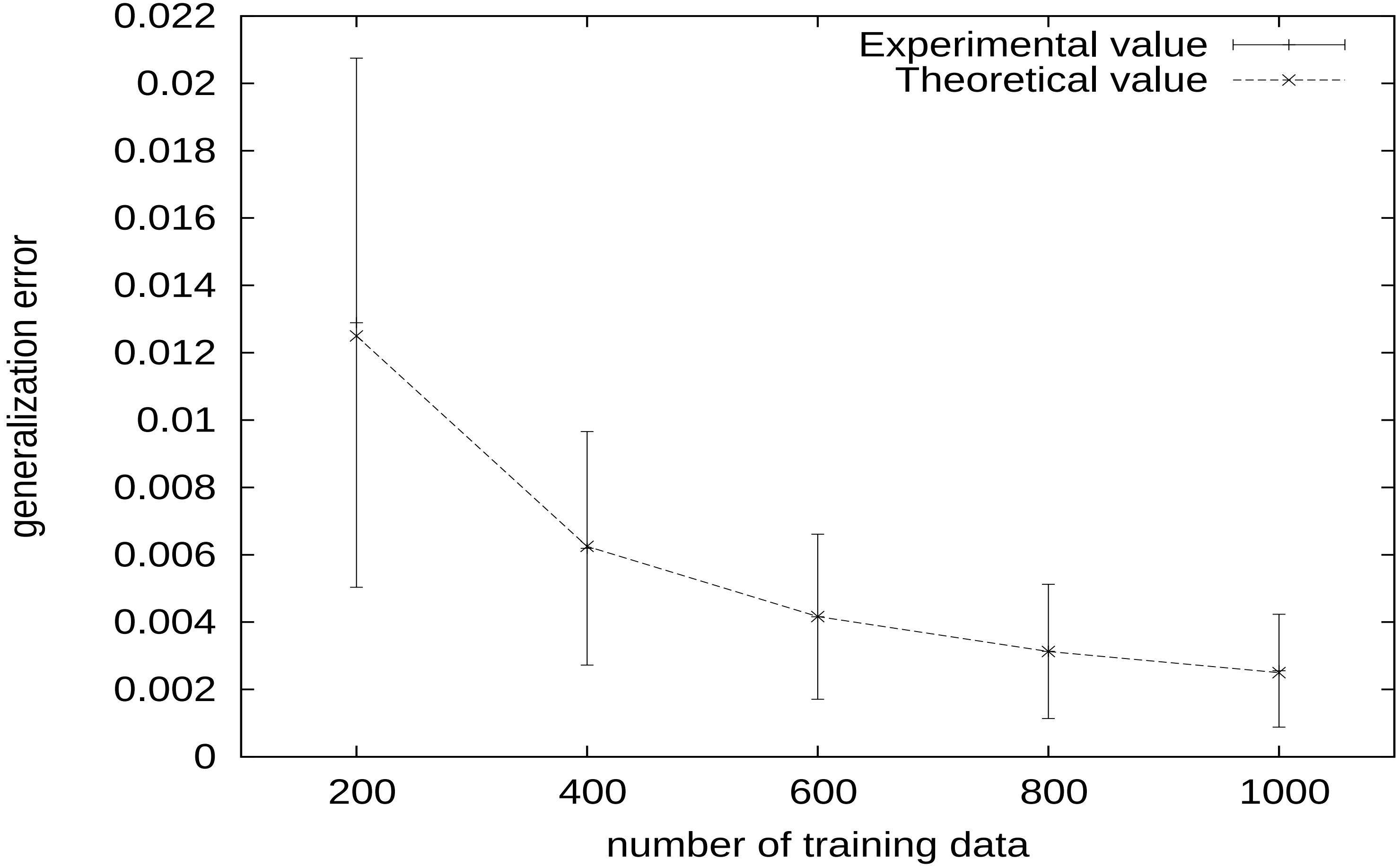
where $i = \max\{j \in \mathbb{Z}\,;\, j^2+j \leq 2(H-(r-1))\}$.

1. $H-r=0: \lambda = \frac{r-1+rN}{2}, \theta = 1.$
2. $H-r=1:$
   (a) If $N>2$, $\lambda = \frac{r(N+1)}{2}, \theta = 1.$
   (b) If $N=2$, $\lambda = \frac{3r}{2}, \theta = 2.$
   (c) If $N=1$, $\lambda = \frac{3}{4}+r-1, \theta = 1.$



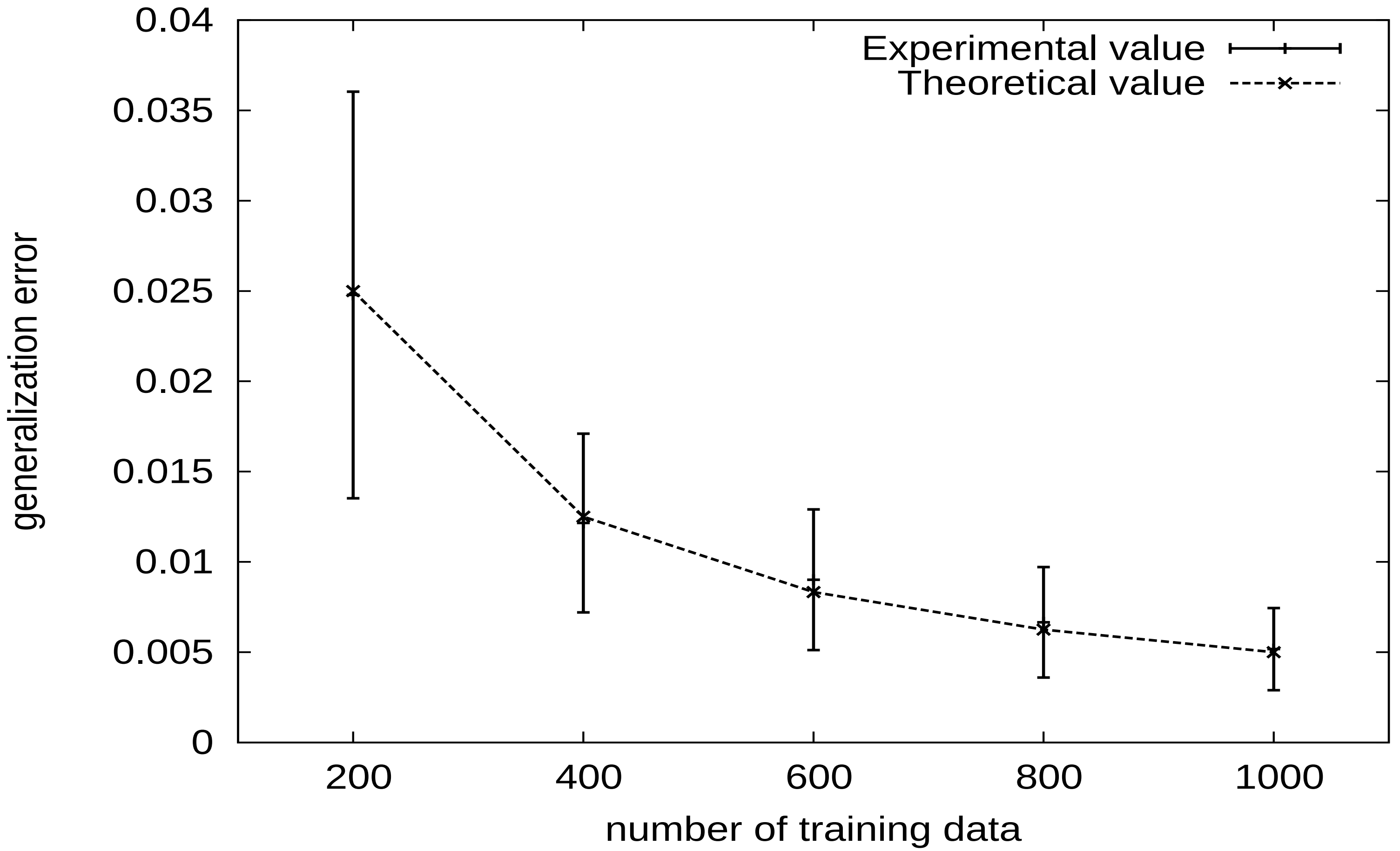
Numerical Results (By K.Nagata)

$N = 4$



Numerical Results (By K.Nagata)

$N = 4$



Lemma 4

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1H} \\ a_{21} & \cdots & a_{2H} \\ \vdots & \vdots & \vdots \\ a_{M1} & \cdots & a_{MH} \end{pmatrix}, Y = \begin{pmatrix} f_{11}(y) & \cdots & f_{1N}(y) \\ f_{21}(y) & \cdots & f_{2N}(y) \\ \vdots & & \vdots \\ f_{H1}(y) & \cdots & f_{HN}(y) \end{pmatrix},$$

where $y = (y_1, \cdots, y_n)$. Then we have the blowing up process of $||AY||^2$ as

$$\sum_{j=1}^{k} \sum_{i=1}^{M} a_{ij}^2 \frac{Y_{jj}^{(j)}}{Y_{j-1,j-1}^{(j)}} + \| \frac{1}{Y_{j-1,j-1}^{(j)}} A^{(k)} Y^{(k)} \|^2,$$

where

Lemma 4

$$A^{(k)} = \begin{pmatrix} a_{1,k+1} & \cdots & a_{1H} \\ & \vdots & \\ a_{M,k+1} & \cdots & a_{MH} \end{pmatrix}, Y^{(k)} = \begin{pmatrix} Y^{(k)}_{k+1,k+1} & \cdots & Y^{(k)}_{k+1,N} \\ \vdots & \vdots & \\ Y^{(k)}_{H,k+1} & \cdots & Y^{(k)}_{HN} \end{pmatrix},$$

$$Y^{(k)}_{ij} = \begin{vmatrix} f_{11}(y) & \cdots & f_{1k}(y) & f_{1j}(y) \\ \vdots & \vdots & & \vdots \\ f_{k1}(y) & \cdots & f_{kk}(y) & f_{1j}(y) \\ f_{i1}(y) & \cdots & f_{ik}(y) & f_{ij}(y) \end{vmatrix},$$

and local analytic coordinate system $(y^{(k)}_1, \cdots, y^{(k)}_n)$ such that $Y^{(k)}_{ij}(y^{(k)})$ are normal crossing.

Lemma 5

$f_1(w_1, \ldots, w_d), \ldots, f_m(w_1, \ldots, w_d)$

: homogeneous functions of $w_1, \cdots, w_d$

with $\deg f_i(w_1, \cdots, w_d) = n_i$.

ψ : C^∞ function with $\psi_{(0,\cdots,0)} \geq \psi_{(w_1^*,\cdots,w_d^*)}$

normal crossing at $(0, \cdots, 0), (w_1^*, \cdots, w_d^*)$.

Then we have

$$c_{(0,\cdots,0)}(\mathbb{R}^d, f_1^2 + \cdots + f_m^2, \psi)$$

$$\leq c_{(w_1^*,\cdots,w_d^*)}(\mathbb{R}^d, f_1^2 + \cdots + f_m^2, \psi),$$

Remark

In general, it is not true that

$$c_{w_0}(\mathbb{R}^d, f_1^2 + \cdots + f_m^2) \leq c_{w^*}(\mathbb{R}^d, f_1^2 + \cdots + f_m^2),$$

where $w_0 \in \mathbb{R}^d$ satisfies

$$f_i(w_0) = \frac{\partial f_i}{\partial w_j}(w_0) = 0, 1 \leq i \leq m, 1 \leq j \leq d.$$

Example

Let $f_1 = x(x-1)^2$,
$f_2 = ((y-1)^6 + x)(y^2 + (x-1)^2)$
$f_3 = ((z-1)^6 + x)(z^2 + (x-1)^2)$.
Then we have

$$f_1 = f_2 = f_3 = \frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} = \frac{\partial f_2}{\partial x} = \frac{\partial f_3}{\partial z} = \frac{\partial f_3}{\partial x} = 0$$

if and only if $x = 1, y = 0, z = 0$.
In this case, we have
$c_{(1,0,0)}(f_1^2 + f_2^2 + f_3^2) = 1/4 + 1/4 + 1/4 >$
$c_{(0,1,1)}(f_1^2 + f_2^2 + f_3^2) = 1/2 + 1/12 + 1/12.$

Lemma 6 Toric Variety

$f(x)$: non-degenerate with respect to its Newton polyhedron Γ_+

If $c = \min\{c' \geq 0 : c'\mathbf{e} \in \Gamma_+\} > 1$ then we have $c_0(f) = 1/c$.

$f_1 = u_1^{s_{11}} \cdots u_d^{s_{1d}}, \cdots, f_p = u_1^{s_{p1}} \cdots u_d^{s_{pd}}$,

$g = u_1^{t_1} u_2^{t_2} \cdots u_d^{t_d} \mathrm{d}u$

Γ_+ : the Newton diagram of $f_1^2 + \cdots + f_p^2$.

$\mathbf{e} = (1, \ldots, 1)^t$, $\mathbf{t} = (t_1, \ldots, t_d)^t$

$c = \min\{c' \geq 0 : c'(\mathbf{t} + \mathbf{e}) \in \Gamma_+\}$

θ' : the number of faces $T \ni c(\mathbf{t} + \mathbf{e})$ with dim. $d - 1$ of Γ_+.

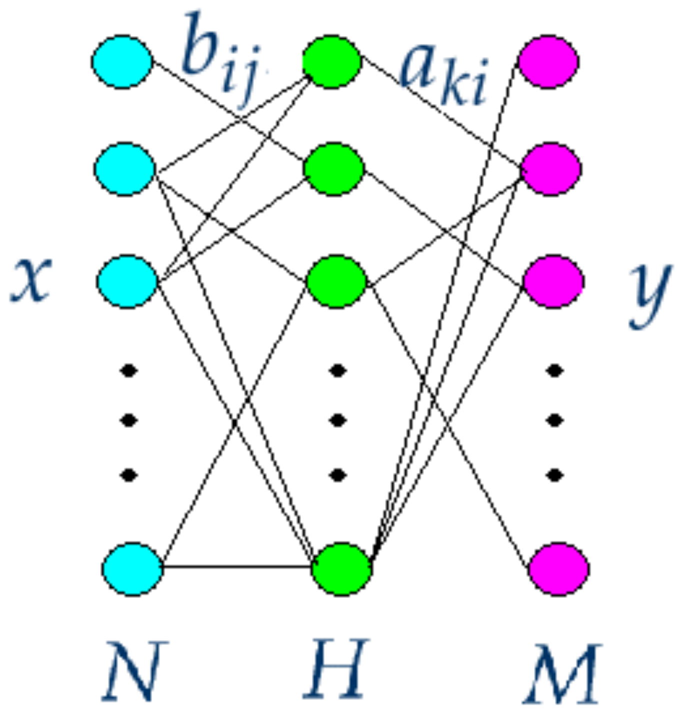
$\theta = \min\{d, \theta'\}$,

Then, the largest pole of $\int_{\text{near } 0} (f_1^2 + \cdots + f_p^2)^z g$ is $1/c$ and its order is θ.

In this case, the condition $c > 1$ is not necessary.

Reduced rank regression

Reduced rank regression. (linear hidden units.)



(1) $N+r \le M+H'$, $M+r \le N+H'$, $H'+r \le M+N$.

(a) If $M+H'+N+r$ is even, then $\theta = 1$ and $\lambda = \frac{-(H'+r)^2-M^2-N^2+2(H'+r)M+2(H'+r)N+2MN}{8}$.

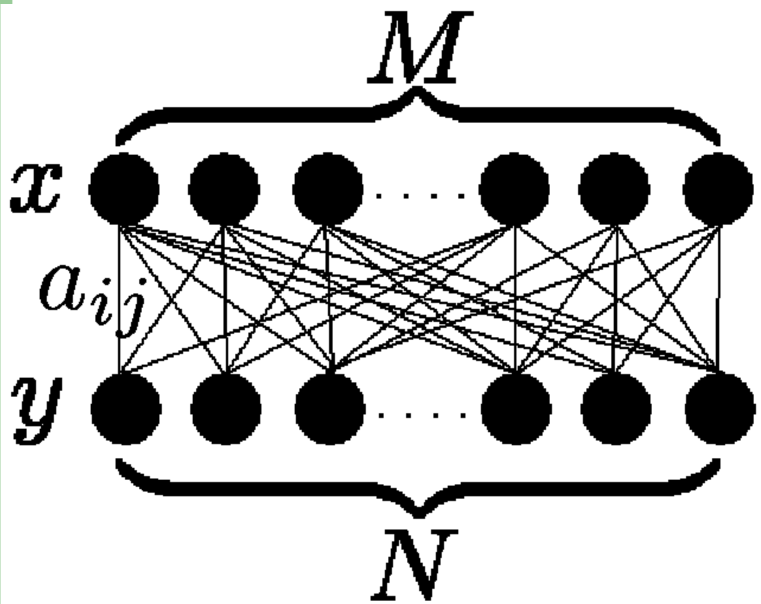
(b) If $M+H'+N+r$ is odd, then $\theta = 2$ and $\lambda = \frac{-(H'+r)^2-M^2-N^2+2(H'+r)M+2(H'+r)N+2MN+1}{8}$.

(2) $M+H' < N+r$ then $\theta = 1$ and $\lambda = \frac{H'M-H'r+Nr}{2}$.

(3) $N+H' < M+r$ then $\theta = 1$ and $\lambda = \frac{H'M-H'r+Nr}{2}$.

(4) $M+N < H'+r$ then $\theta = 1$ and $\lambda = \frac{MN}{2}$.

Restricted Boltzmann Machine



$x = (x_j) \in \{1, -1\}^M$ inputs

$y = (y_j) \in \{1, -1\}^N$ hidden units

$$p(x, y|a) = \frac{\exp(\sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} x_i y_j)}{Z(a)}$$

$$p(x|a) = \sum_{y_i = \pm 1} p(x, y|a) = \frac{\prod_{j=1}^{N} \prod_{i=1}^{M} \cosh(a_{ij})}{Z(a)}$$

$$\times \prod_{j=1}^{N} (2 \sum_{0 \le p \le M/2} \sum_{i_1 < \cdots < i_{2p}} x_{i_1} \cdots x_{i_{2p}} \tanh(a_{i_1 j}) \cdots \tanh(a_{i_{2p} j}))$$

Restricted Boltzmann Machine

$$\prod_{j=1}^{N}\Big(\sum_{0\le p\le M/2}\ \sum_{i_1<\cdots<i_{2p}} x_{i_1}\cdots x_{i_{2p}}\tanh(a_{i_1 j})\cdots\tanh(a_{i_{2p} j})\Big)$$

$$= \sum_{I\in\mathcal{I}} B_N^I x^I, \quad B=(b_{ij})=(\tanh(a_{ij})) : M\times N \text{ matrix} .$$

Let $B=(b_{ij})=(\tanh(a_{ij}))$. Denote $B^J=\prod_{i=1}^M\prod_{j=1}^N b_{ij}^{J_{ij}}$ where $J=(J_{ij})$ is an $M\times N$ matrix with $J_{ij}\in\{0,1\}$.

$\mathcal{I}=\{I\in\{0,1\}^M \mid \sum_{i=1}^M I_i \text{ is even }\}$,

$B^I=\sum_{\substack{\sum_{i=1}^M J_{ij}=0 \mod 2\\ \sum_{j=1}^N J_{ij}=0 \mod I_i}} B^J$ for $I\in\mathcal{I}$.

Consider $c_{B^*}\left(\sum_{I\in\mathcal{I}}\left(\frac{B_N^I}{{B^*}_N^I}-\frac{B_N^0}{{B^*}_N^0}\right)^2\right)$

Example

Example (M=3), $I = (0,0,0), (1,1,0), (1,0,1)$ or $(0,1,1)$.

$$C_i = \begin{pmatrix} 1 & b_{1i}b_{2i} & b_{1i}b_{3i} & b_{2i}b_{3i} \\ b_{1i}b_{2i} & 1 & b_{2i}b_{3i} & b_{1i}b_{3i} \\ b_{1i}b_{3i} & b_{2i}b_{3i} & 1 & b_{1i}b_{2i} \\ b_{2i}b_{3i} & b_{1i}b_{3i} & b_{1i}b_{2i} & 1 \end{pmatrix}, \begin{pmatrix} B_N^{(0,0,0)} \\ B_N^{(1,1,0)} \\ B_N^{(1,0,1)} \\ B_N^{(0,1,1)} \end{pmatrix} = (\prod_{i=1}^{N} C_i) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Theorem

Consider the matrix $C_N = (c_N^{I,I'})$ where $c_N^{I,I'} = b_N^{I''}$ with $I' + I'' = I \mod 2$.
Note that $B_N = C_N B_{N-1}$.

Theorem
Let $K_1 \cap K_2 = \phi$ and $K_1 \cup K_2 = \{1, \ldots, M\}$.

Set $\ell_I = \begin{cases} -1, & \text{if } \#\{i \in K_1 : I_i = 1\} \text{ is odd}, \\ 1, & otherwise. \end{cases}$

Then $\ell = (\ell_I)$ is an eigenvector of C_N and its eigenvalue is $\sum_{I \in \mathcal{I}} \ell_I b_N^I$.

Example $(M = 3)$

$$C_j = \begin{pmatrix} 1 & b_{1j}b_{2j} & b_{1j}b_{3j} & b_{2j}b_{3j} \\ b_{1j}b_{2j} & 1 & b_{2j}b_{3j} & b_{1j}b_{3j} \\ b_{1j}b_{3j} & b_{2j}b_{3j} & 1 & b_{1j}b_{2j} \\ b_{2j}b_{3j} & b_{1j}b_{3j} & b_{1j}b_{2j} & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$B_N = \begin{pmatrix} B_N^{(0,0,0)} \\ B_N^{(1,1,0)} \\ B_N^{(1,0,1)} \\ B_N^{(0,1,1)} \end{pmatrix} = C_N C_{N-1} C_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, DC_jD^{-1} = \begin{pmatrix} s_j^0 & 0 & 0 & 0 \\ 0 & s_j^1 & 0 & 0 \\ 0 & 0 & s_j^2 & 0 \\ 0 & 0 & 0 & s_j^3 \end{pmatrix}$$

$$\begin{cases} s_j^0 = 1 + b_{1j}b_{2j} + b_{1j}b_{3j} + b_{2j}b_{3j}, \quad s_j^1 = 1 + b_{1j}b_{2j} - b_{1j}b_{3j} - b_{2j}b_{3j}, \\ s_j^2 = 1 - b_{1j}b_{2j} + b_{1j}b_{3j} - b_{2j}b_{3j}, \quad s_j^3 = 1 - b_{1j}b_{2j} - b_{1j}b_{3j} + b_{2j}b_{3j}, \end{cases}$$

Main Theorem

(1) If $M = 2$ then

$$\lambda = \frac{1}{2} \text{ and } \theta = \begin{cases} 2, & \text{if } N = 1, b^* = 0 \\ 1, & \text{otherwise.} \end{cases}$$

(2) If $M = 3$ then

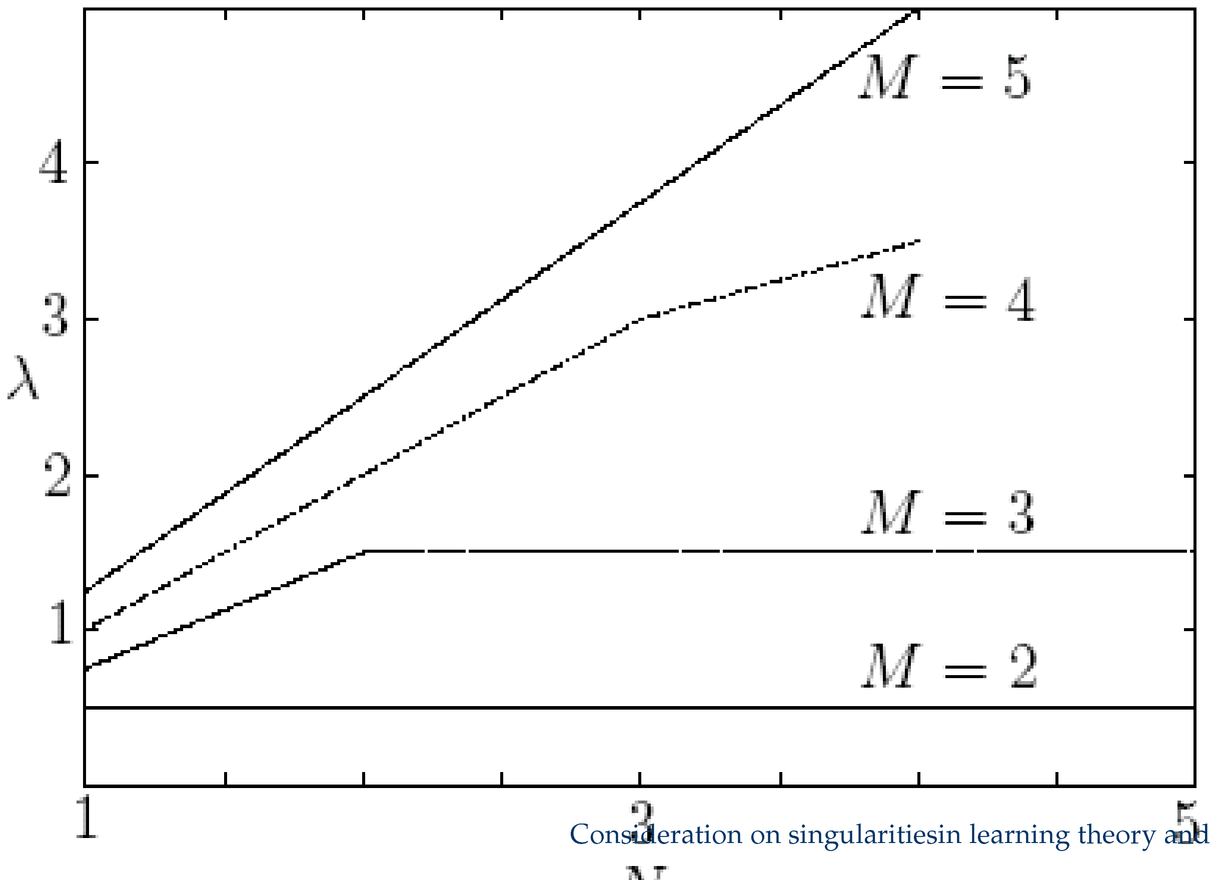
$$\lambda = \begin{cases} 3/4, & \text{if } N = 1, b^* = 0 \\ 1/2, & \text{if } N = 1, b^* \neq 0, \prod_{i=1}^{3} b^*_{i1} = 0 \\ 3/2, & \text{if } N = 1, \prod_{i=1}^{3} b^*_{i1} \neq 0 \\ 3/2, & \text{if } N \geq 2, \end{cases}$$

$$\theta = \begin{cases} 3, & \text{if } N = 2, b^* = 0, \\ 2, & \text{if } N = 2, b^* \neq 0, \\ & b^*_{i_0 j} = b^*_{i_1 j} = 0 \\ 2, & \text{if } N = 2, b^*_{i_0 j_0} b^*_{i_1 j_0} \neq 0, \\ & b^*_{i_2 j_0} = b^*_{ij} = 0 \text{ for } 1 \leq i \leq 3 \\ 1, & \text{otherwise ,} \end{cases}$$

$i_0, i_1, i_2 \in \{1, 2, 3\}$ are different from each other, $1 \leq j_0 \leq N$.

Main Theorem 2

If $M > N$ and $b^* = 0$ then $\lambda = \dfrac{MN}{4}$ and $\theta = \begin{cases} 1, & \text{if } M > N+1, \\ M, & \text{if } M = N+1. \end{cases}$



Main Theorem 2

$(b_{1j}, b_{2j}, \cdots, b_{Mj}) \neq 0$ for $j = 1, \dots, N_0$
$(b_{1j}, b_{2j}, \cdots, b_{Mj}) = 0$ for $j = N_0 + 1, \dots, N$ in V,
where V is a sufficiently small neighborhood of b^*.

Then

$$\frac{M(N-N_0)}{4} \leq \lambda \leq \frac{M(N-N_0)}{4} + \frac{MN_0}{2},$$
$$\text{if } M > N - N_0$$
$$\frac{M(M-1)}{4} + \frac{MN_0}{2} \leq \lambda \leq \frac{2N_0 + (M-1)(M-2)}{4} + \frac{MN_0}{2}$$
$$\left(< \frac{MN_0}{2} + \frac{M(N-N_0)}{4} \right),$$
$$\text{if } M \leq N - N_0.$$

Conclusion

Conclusion

- We show new bound values of learning coefficients for Vandermonde matrix type singularities,
- and the explicit values in some conditions.
- Our future research aims to improve our methods and to apply them to obtain exact values.

Proof

$$\left\langle \begin{pmatrix} b_1^m \\ \vdots \\ b_H^m \end{pmatrix}, \begin{pmatrix} 0 \\ b_2^m(b_1^Q - b_2^Q) \\ \vdots \\ b_H^m(b_1^Q - b_H^Q) \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_1^m(b_1^Q - b_H^Q)\cdots(b_{H-1}^Q - b_H^Q) \end{pmatrix} \right\rangle$$

$= < \mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_H > .$

We have $\langle \mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_H \rangle = \langle \mathbf{b}'_1, \mathbf{b}'_2, \ldots, \mathbf{b}'_{H-1}, \mathbf{b}''_H \rangle$.

Let $g_{j,j}(x), \ldots, g_{H,j}(x)$ s.t. $g_{j',j}(x\gamma_{j'}) = g_{j'',j}(x\gamma_{j''})$ if $|b^*_{j'}| = |b^*_{j''}| \neq 0$ and $g_{j',j}(x) - g_{j'',j}(x')$ can be devided by $x^Q - x'^Q$ if $b^*_{j'} = b^*_{j''} = 0$

Assume that $(0, \cdots, 0, g_{j,j}(b_j)\mathbf{b}''_{jj}, \cdots, g_{H,j}(b_H)\mathbf{b}''_{Hj}) \in \langle \mathbf{b}''_j, \ldots, \mathbf{b}''_H \rangle$

Proof

Since

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_{j-1}^m (b_1^Q - b_{j-1}^Q) \cdots (b_{j-2}^Q - b_{j-1}^Q) \\ \vdots \\ b_H^m (b_1^Q - b_H^Q) \cdots (b_{j-2}^Q - b_H^Q) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g_{j-1,j-1}(b_{j-1}) \mathbf{b}''_{j-1,j-1} \\ \vdots \\ g_{H,j-1}(b_H) \mathbf{b}''_{H,j-1} \end{pmatrix},$$

Proof

we have $\mathbf{b}'_{j-1} =$

$$\mathbf{b}''_{j-1} g_{j-1,j-1}(b_{j-1}) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (g_{j,j-1}(b_j) - g_{j-1,j-1}(b_{j-1}))\mathbf{b}''_{j,j-1} \\ \vdots \\ (g_{H,j-1}(b_H) - g_{j-1,j-1}(b_{j-1}))\mathbf{b}''_{H,j-1} \end{pmatrix}$$

Proof

$$= \mathbf{b}''_{j-1} g_{j-1,j-1}(b_{j-1}) + (0, \cdots, 0, g_{j,j}(b_j)\mathbf{b}''_{j,j}, \cdots, g_{H,j}(b_H)\mathbf{b}''_{H,j})^t,$$

where

$$\begin{cases} g_{k,j}(b_k) = g_{k,j-1}(b_k) - g_{j-1,j-1}(b_{j-1}), \\ \qquad\qquad \text{if } |b^*_k| \neq |b^*_{j-1}|, \\ g_{k,j}(b_k) = (g_{k,j-1}(b_k) - g_{j-1,j-1}(b_{j-1}))/(b_{j-1}/\gamma_{j-1} - b_k/\gamma_k), \\ \qquad\qquad \text{if } |b^*_k| = |b^*_{j-1}| \neq 0, \\ g_{k,j}(b_k) = (g_{k,j-1}(b_k) - g_{j-1,j-1}(b_{j-1}))/(b^Q_{j-1} - b^Q_k) \\ \qquad\qquad \text{if } b^*_k = b^*_{j-1} = 0. \end{cases}$$

By the inductive assumption,

$$(0, \cdots, 0, g_{j,j}(b_j)\mathbf{b}''_{j,j}, \cdots, g_{H,j}(b_H)\mathbf{b}''_{H,j})^t \in < \mathbf{b}''_j, \cdots, \mathbf{b}''_H >.$$

$$\langle \mathbf{b}'_1, \cdots, \mathbf{b}'_{j-1}, \mathbf{b}''_j, \cdots, \mathbf{b}''_H \rangle = \langle \mathbf{b}'_1, \cdots, \mathbf{b}'_{j-2}, \mathbf{b}''_{j-1}, \mathbf{b}''_j, \cdots, \mathbf{b}''_H \rangle.$$

Lemma 例

$b_1^* = 1, b_2^* = 1, b_3^* = 2, b_4^* = 2$

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_2(b_1^2 - b_2^2) & 0 & 0 \\ b_3 & b_3(b_1^2 - b_3^2) & b_3(b_1^2 - b_3^2)(b_2^2 - b_3^2) & 0 \\ b_4 & b_4(b_1^2 - b_4^2) & b_4(b_1^2 - b_4^2)(b_2^2 - b_4^2) & b_4(b_1^2 - b_4^2)(b_2^2 - b_4^2)(b_3^2 - b_4^2) \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_2(b_1^2 - b_2^2) & 0 & 0 \\ b_3 & b_3(b_1^2 - b_3^2) & b_3(b_1^2 - b_3^2)(b_2^2 - b_3^2) & 0 \\ b_4 & b_4(b_1^2 - b_4^2) & b_4(b_1^2 - b_4^2)(b_2^2 - b_4^2) & (b_3 - b_4) \end{pmatrix} R_1$$

Lemma 例

$$\begin{pmatrix} b_3(b_2^2-b_3^2)(b_1^2-b_3^2) & 0 \\ b_3(b_2^2-b_3^2)(b_1^2-b_3^2)+b_4(b_1^2-b_4^2)(b_2^2-b_4^2)-b_3(b_1^2-b_3^2)(b_2^2-b_3^2) & (b_3-b_4) \end{pmatrix}$$

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_2(b_1^2-b_2^2) & 0 & 0 \\ b_3 & b_3(b_1^2-b_3^2) & b_3(b_2^2-b_3^2)(b_1^2-b_3^2) & 0 \\ b_4 & b_4(b_1^2-b_4^2) & b_3(b_2^2-b_3^2)(b_1^2-b_3^2) & (b_3-b_4) \end{pmatrix} R_3$$

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_2(b_1+b_2)(b_1-b_2) & 0 & 0 \\ b_3 & b_3(b_1^2-b_3^2) & 1 & 0 \\ b_4 & b_4(b_1^2-b_4^2) & 1 & (b_3-b_4) \end{pmatrix} R_4$$

Lemma 例

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & (b_1-b_2) & 0 & 0 \\ b_3 & (b_1-b_2)+b_3(b_1^2-b_3^2)-(b_1-b_2) & 1 & 0 \\ b_4 & (b_1-b_2)+b_4(b_1^2-b_4^2)-(b_1-b_2) & 1 & (b_3-b_4) \end{pmatrix} R_5$$

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & (b_1-b_2) & 0 & 0 \\ b_3 & (b_1-b_2) & 1 & 0 \\ b_4 & (b_1-b_2) & 1 & (b_3-b_4) \end{pmatrix} R_6$$

Lemma 例

$$\begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_2 & b_1 - b_2 & 0 & 0 \\ b_3 & 0 & 1 & 0 \\ b_3 & 0 & 1 & b_3 - b_4 \end{pmatrix} R_7 = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ b_1 & b_1 - b_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & b_3 - b_4 \end{pmatrix} R_8$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & b_1 - b_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & b_3 - b_4 \end{pmatrix} R_9$$

Lemma

Let $\{|b_0^{**}|, \ldots, |b_r^{**}|\ ;\ |b_i^*|, i = 1, \cdots, H\}$.
Assume that $b_1^* = \cdots = b_{H_0}^* = b_0^{**}$, $|b_{H_0+1}^*| = \cdots = |b_{H_0+H_1}^*| = |b_1^{**}|$,
$\ldots$, $|b_{H_0+\cdots+H_{r-1}+1}^*| = \cdots = |b_{H_0+\cdots+H_r}^*| = |b_r^{**}|$.

$$\exists R \text{ s.t. } B'R = \begin{pmatrix} B^{(0)} & 0 & 0 & \cdots & 0 \\ 0 & B^{(1)} & 0 & \cdots & 0 \\ & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & B^{(r)} \end{pmatrix}, \text{ where}$$

$$B^{(0)} = \begin{pmatrix} {b_1^{(0)}}^m & \cdots & {b_1^{(0)}}^{Q(H_0-1)+m} \\ \vdots & & \vdots \\ {b_{H_0}^{(0)}}^m & \cdots & {b_{H_0}^{(0)}}^{Q(H_0-1)+m} \end{pmatrix}, B^{(\alpha)} = \begin{pmatrix} 1 & b_1^{(\alpha)} & \cdots & {b_1^{(\alpha)}}^{H_\alpha-1} \\ \vdots & & \vdots & \\ 1 & b_{H_\alpha}^{(\alpha)} & \cdots & {b_{H_\alpha}^{(\alpha)}}^{H_\alpha-1} \end{pmatrix}$$

Proof

Since

$$\begin{pmatrix} \prod_{j=1}^{N} {b_{1j}}^{\ell_j} \\ \prod_{j=1}^{N} {b_{2j}}^{\ell_j} \\ \vdots \\ \prod_{j=1}^{N} {b_{Hj}}^{\ell_j} \end{pmatrix} = \begin{pmatrix} {b_{11}}^{\ell'_1} \prod_{j=2}^{N} {b_{1j}}^{\ell_j} & 0 & \cdots & 0 \\ 0 & {b_{21}}^{\ell'_1} \prod_{j=2}^{N} {b_{2j}}^{\ell_j} & \cdots & 0 \\ \vdots & \ddots & & 0 \\ 0 & 0 & \cdots & {b_{H1}}^{\ell'_1} \prod_{j=2}^{N} {b_{Hj}}^{\ell_j} \end{pmatrix}$$

$$\times \begin{pmatrix} {b_{11}}^{\ell_1-\ell'_1} \\ {b_{21}}^{\ell_1-\ell'_1} \\ \vdots \\ {b_{H1}}^{\ell_1-\ell'_1} \end{pmatrix}$$, we have Theorem 1.

Theorem 2 の証明

$$\Psi' dw = \{\prod_{\tau=0}^{r} v_\tau^{t_\tau}\left(d_1^2 + d_2^2 + \cdots + d_K^2\right) + \sum_{i=K+1}^{p'} C_i^2\}^z \prod_{\tau=0}^{r} v_\tau^{q_\tau} \mathrm{d}d\mathrm{d}v\mathrm{d}a\mathrm{d}e_m$$

$$C_i = \prod_{\tau=0}^{r} v_\tau^{t(i,0,\tau)} \sum_{\substack{i\le m\le p' \\ J_m^{(\alpha)}=0}} g(i,m) a_m e_m \prod_{\substack{k+1\le m'<i \\ J_{m'}^{(\alpha)}=0}} (e_m^Q - e_{m'}^Q)$$

$$+ \sum_{J\in RJ^{(\alpha)}} \prod_{\tau=0}^{r} v_\tau^{t(i,J,\tau)} \sum_{\substack{i\le m\le p' \\ J_m^{(\alpha)}=J}} g(i,m) a_m e_m \prod_{\substack{k+1\le m'<i \\ J_{m'}^{(\alpha)}=J}} (e_m - e_{m'})$$

$$+ \sum_{J\notin RJ^{(\alpha)}, J\neq 0} \prod_{\tau=0}^{r} v_\tau^{t(i,J,\tau)} \sum_{\substack{i\le m\le p' \\ J_m^{(\alpha)}=J}} g(i,m) a_m \prod_{\substack{k+1\le m'<i \\ J_{m'}^{(\alpha)}=J}} (e_m - e_{m'}).$$

Theorem 2 の証明

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1H} \\ a_{21} & a_{22} & \cdots & a_{2H} \\ & & \vdots & \\ a_{M1} & a_{M2} & \cdots & a_{MH} \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} b_1^m & 0 & 0 & \cdots & 0 \\ 0 & b_2^m(b_2^Q - b_1^Q) & 0 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & b_H^m(b_H^Q - b_1^Q)(b_H^Q - b_2^Q)\cdots(b_H^Q - b_{H-1}^Q) \end{pmatrix}$$

の log canonical threshold と同じ

Normal mixture model

$$\begin{pmatrix} a_1 \\ \vdots \\ a_H \\ a^*_{H+1} \\ \vdots \\ a^*_{H+r} \end{pmatrix}^t \begin{pmatrix} \prod_{j=1}^N b_{1j}^{\ell_j} \\ \vdots \\ \prod_{j=1}^N b_{H-1,j}^{\ell_j} \\ \prod_{j=1}^N b_{Hj}^{\ell_j} \\ \prod_{j=1}^N {b^*_{H+1,j}}^{\ell_j} \\ \vdots \\ \prod_{j=1}^N {b^*_{H+r,j}}^{\ell_j} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{H-1} \\ a^*_{H+1} \\ \vdots \\ a^*_{H+r} \end{pmatrix}^t \begin{pmatrix} \prod_{j=1}^N b_{1j}^{\ell_j} - \prod_{j=1}^N b_{Hj}^{\ell_j} \\ \prod_{j=1}^N b_{2j}^{\ell_j} - \prod_{j=1}^N b_{Hj}^{\ell_j} \\ \vdots \\ \prod_{j=1}^N b_{H-1,j}^{\ell_j} - \prod_{j=1}^N b_{Hj}^{\ell_j} \\ \prod_{j=1}^N {b^*_{H+1,j}}^{\ell_j} - \prod_{j=1}^N b_{Hj}^{\ell_j} \\ \vdots \\ \prod_{j=1}^N {b^*_{H+r,j}}^{\ell_j} - \prod_{j=1}^N b_{Hj}^{\ell_j} \end{pmatrix}$$

by using $\sum_{i=1}^H a_i = 1, \sum_{i=1}^r a^*_{H+i} = -1.$

$$(\prod_{j=1}^N (b_{1j} - b_{Hj})^{\ell_j}, \cdots, \prod_{j=1}^N (b_{H-1,j} - b_{Hj})^{\ell_j}, \prod_{j=1}^N (b^*_{H+1,j} - b_{Hj})^{\ell_j}, \cdots, \prod_{j=1}^N (b^*_{H+r,j} - b_{Hj})^{\ell_j})$$