# Computational algebraic geometry

## Learning coefficients via symbolic and numerical methods

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# Ideals $\leftrightarrow$ varieties

- Let $k$ be a field ($\mathbb{R}$ or $\mathbb{C}$).
- Ideal in $R = k[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_r$ is

$$I = \langle f_1, \ldots, f_r \rangle = \sum_{i=1}^{r} g_i f_i \subset R$$

- Variety defined by $I$ is

$$V = \mathbb{V}(I) = \{p \in k^n : \forall f \in I,\ f(p) = 0\}$$

- The vanishing ideal of a variety $V$ is

$$I = \mathbb{I}(V) = \{f \in R : f(V) = 0\}$$

- The radical of $I$ is

$$\sqrt{I} = \{g : \exists m,\ g^m \in I\} \subset R, \quad \mathbb{V}(I) = \mathbb{V}(\sqrt{I})$$

  $I$ is a radical ideal if $I = \sqrt{I}$
- Hilbert Nullstellensatz: if $k = \mathbb{C}$ then $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$ for all $J \subset R$.

# Gröbner bases, invariants

- A monomial order $>$ is given by weights $w_1, w_2, \ldots \in \mathbb{R}^n_{\geq 0}$:

$$x^\alpha > x^\beta \Leftrightarrow \begin{array}{l} w_1 \cdot \alpha > w_1 \cdot \beta \text{ or} \\ w_2 \cdot \alpha > w_2 \cdot \beta \text{ or} \\ \quad \ldots \end{array}$$

  such that $>$ is total.
  Example: $>_{lex}$ is given by $w_1 = e_1, w_2 = e_2, \ldots$.
- The initial term of $f \in R$ is the term (coefficient·monomial) with the largest monomial occurring with a nonzero coefficient:

  e.g. for $f = 3x^2y + 4xy^7 + 2$ it is $\mathrm{in}_{lex}(f) = 3x^2y$.
- A Gröbner basis of $I$ is a set of generators $G$ of $I$ such that

$$\langle \mathrm{in}(G) \rangle = \langle \mathrm{in}(I) \rangle.$$

- Invariants such as dimension, degree, Hilbert polynomial can be computed using Gröbner bases.

# Computational (symbolic) algebraic geometry

- Cox, Little, O'Shea, Ideals, varieties, and algorithms
- Cox, Little, O'Shea, Using algebraic geometry
- Greuel, Pfister, A Singular Introduction to Commutative Algebra

Software:

- Singular (can resolve singularities)
- Macaulay2 (can do everything else)
- Magma (Australia), CoCoA (Italy), risa/asir (Japan), ... (other specialized software)
- Sage, Maple, Mathematica, ... (more general software)

# Polynomial homotopy continuation

- Target system: $n$ equations in $n$ variables,

$$F(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) = \mathbf{0},$$

  where $f_i \in R = \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, ..., x_n]$ for $i = 1, ..., n$.
- Start system: $n$ equations in $n$ variables:

$$G(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) = \mathbf{0},$$

  such that it is easy to solve.
- Homotopy: for $\gamma \in \mathbb{C} \setminus \{0\}$ consider

$$H(\mathbf{x}, t) = (1-t)G(\mathbf{x}) + \gamma t F(\mathbf{x}),\ t \in [0, 1].$$

# Example

target

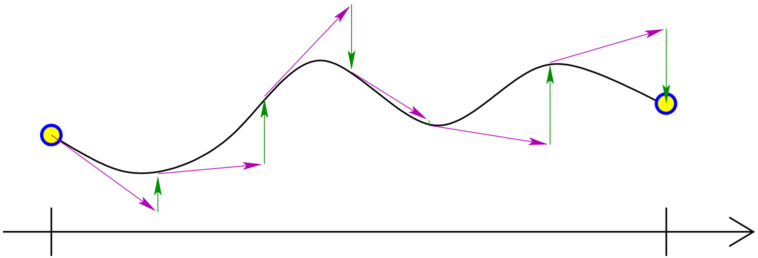
$$f_1 = x_1^4 x_2 + 5x_1^2 x_2^3 + x_1^3 - 4$$
$$f_2 = x_1^2 - x_1 x_2 + x_2 - 8$$

start

$$g_1 = x_1^5 - 1$$
$$g_2 = x_2^2 - 1$$

Start solutions → target solutions:

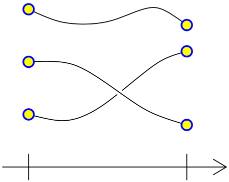
$$H(\mathbf{x}, t) = 0 \text{ implies } \frac{d\mathbf{x}}{dt} = -\left(\frac{\partial H}{\partial \mathbf{x}}\right)^{-1} \frac{\partial H}{\partial t}.$$



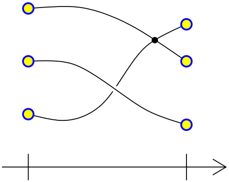
# Global picture

Optimal homotopy:

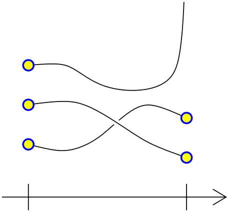
- the continuation paths are regular;
- the homotopy establishes a bijection between the start and target solutions.



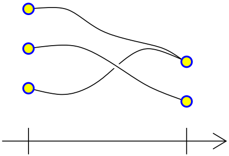
Possible singular scenarios:



non-generic



diverging paths



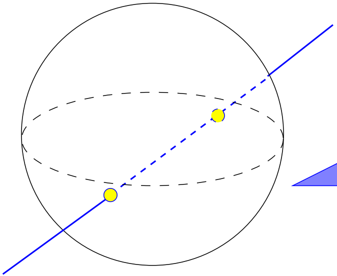
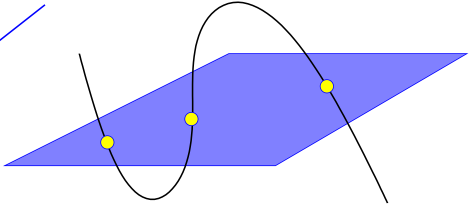
multiple solutions

# Higher-dimensional solution sets

- Let $I = (f_1, \ldots, f_N)$ be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$.
- Goal: Understand the variety

$$X = \mathbb{V}(I) = \{\mathbf{x} \in \mathbb{C}^n \mid \forall f \in I,\ f(\mathbf{x}) = 0\}.$$

- A witness set for an equidimensional component $Y$ of $X$:
  - a generic “slicing” plane $L$ with $\dim L = \operatorname{codim} Y$
  - witness points $w_{Y,L} = Y \cap L$
  - (generators of $I$)

# Numerical algebraic geometry

- Sommese, Verschelde, and Wampler, Introduction to Numerical AG (2005)
- Sommese and Wampler, The numerical solution of systems of polynomials (2005)

Software:

- PHCpack (Verschelde);
- HOM4PS (group of T.Y.Li);
- Bertini (group of Sommese);
- NAG4M2: Numerical Algebraic Geometry for Macaulay2 (L.).

and more, e.g.: Maple's ROOTFINDING[HOMOTOPY].

# Analysis

- For a polynomial $f$, the function $|f|^s$ is locally integrable $\mathrm{Re}\, s > 0$.
- Hence, $|f|^s$ is a generalized function defined on $\{s : \mathrm{Re}\, s > 0\} \subset \mathbb{C}$: for a distribution $\varphi \in C_c^\infty$,

$$|f|^s(\varphi) = \int |f(x)|^s \varphi(x)\, dx$$

- Gelfand [1957]: Does it extend to a meromorphic function on $\mathbb{C}$?
- I. N. Bernstein [1968]: Yes. The poles are contained in a finite number of arithmetic progressions.
- Key ingredients: resolution of singularities and being able to write a functional equation

$$b(s) f^s = P \cdot f^{s+1}$$

when $f$ is a monomial. (Here: $b(s)$ is a univariate polynomial and $P$ is a linear differential operator with coefficients in $\mathbb{C}[\boldsymbol{x}, s]$.)

# Invariants in singularity theory

## Definition (Multiplier ideal for $\boldsymbol{f} = (f_1, \ldots, f_r)$)

$$\mathcal{J}(\boldsymbol{f}^c) = \left\{ h \in \mathbb{C}[\boldsymbol{x}] : \frac{|h|^2}{(\sum |f_i|^2)^c} \text{ is locally integrable } \right\}.$$

For $r = 1$, it is the ideal of $h$, that make $\frac{|h|}{|f_1|^c}$ locally integrable.

- Algebrao-geometric definition: via log-canonical resolutions.
- Jumping coefficients of $\boldsymbol{f}$: rational numbers

$$0 = \xi_0 < \xi_1 < \xi_2 < \cdots$$

  such that $\mathcal{J}(\boldsymbol{f}^c)$ is constant exactly for $c \in [\xi_i, \xi_{i+1})$.
- $\xi_1$ is called the log-canonical threshold.
- These invariants measure singularities of the corresponding variety; in particular, they depend only on the ideal $\langle \boldsymbol{f} \rangle$.

# Weyl algebra

- Let $K$ be a field of characteristic zero. (Think: $K = \mathbb{C}$)
- Affine space: $X = K^n$.
- Weyl algebra: an associative algebra

$$D_X = K\langle \boldsymbol{x}, \boldsymbol{\partial} \rangle = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

  where $[\partial_i, x_i] = \partial_i x_i - x_i \partial_i = 1$ and all other pairs of generators commute.
- $D_X$ is isomorphic to the algebra of linear differential operators with polynomial coefficients.
- Every element has the normal form

$$Q = \sum_{\alpha, \beta \in \mathbb{Z}^n} c_{\alpha\beta} \boldsymbol{x}^{\alpha} \boldsymbol{\partial}^{\beta},$$

  where finitely many of $c_{\alpha\beta} \in K$ are nonzero.

# $D$-modules

- $D_X$ is simple: only trivial two-sided ideals.
- We consider only left ideals and left $D_X$-modules.
- Examples of $D$-modules: $K[\boldsymbol{x}]$, $K[[\boldsymbol{x}]]$, $C^\infty(X)$.
- Software:
  - `kan/sm1` (Takayama)
  - `risa/asir` (Noro)
  - `dmod.lib, Singular` (Levandovsky et al.)
  - `D-modules, Macaulay2` (L., Tsai)

# Gröbner bases

- $D_X$ is Gröbner-friendly: $D_X$ is an algebra of solvable type.
- Gröbner bases can be computed with respect to any $w$-compatible monomial order, where $w = (w_{\boldsymbol{x}}, w_{\boldsymbol{\partial}}) \in \mathbb{R}^{2n}$ satisfies $w_{\boldsymbol{x}} + w_{\boldsymbol{\partial}} \geq 0$ componentwise.
- The Bernstein-Sato polynomial $b_f(s) \neq 0$ is the monic polynomial $b(s)$ of the minimal degree satisfying

$$b(s)f^s = P \cdot f^{s+1} = Pf \cdot f^s \in D_X[s]f^s,$$

  where $P \in D_X[s]$ and $D_X[s]f^s$ is a cyclic $D_X[s]$-module generated by $f^s$.
- One can computed $b_f(s)$ via Gröbner bases.

# Connection between $\operatorname{lct}(f)$ and $b_f(s)$

- Assume $k = \mathbb{C}$ then
  - The log-canonical threshold $c_0$ is the lowest root of $b_{\boldsymbol{f}}(-s)$.
  - Every jumping coefficient $c \in [c_0, c_0 + 1)$ is a root of $b_{\boldsymbol{f}}(-s)$.
- For $k = \mathbb{R}$ there are examples where $\operatorname{rlct}(f)$ is not a jumping coefficient of $f$.
  However, $\operatorname{rlct}(f) \mod 1$ equals some root of $b_{\boldsymbol{f}}(-s)$.
- M2 packages where $\operatorname{lct}$ is computed:
  - `Dmodules` (general case)
  - `MonomialMultiplierIdeals` (monomial ideals)
  - `HyperplaneArrangements` (central hyperplane arrangements)

# Improper integrals

To find $c = \text{rlct}(f)$ look for the pole $s = -c$ closest to 0 of

$$|f|^s(\varphi) = \int_{x \in \mathbb{R}^n} |f(x)|^s \varphi(x)\, dx.$$

Idea:

- Pick $\varphi$ supported on a neighborhood of a singularity of the variety $\{f = 0\}$ and evaluate $|f|^s(\varphi)$ numerically.
- If $|f|^{-c_1}(\varphi) < \infty$ and $|f|^{-c_2}(\varphi) = \infty$ then $c_1 < \text{rlct}(f) \leq c_2$.
- We can determine possible values of $\text{rlct}(f)$ by looking at $b_f(s)$?

# Questions

- Can we say something about the denominator of $\mathrm{rlct}(f)$?
- How to compute an improper integral numerically?
  How to determine it equals $\infty$?
- Can we use statistics to estimate $\mathrm{rlct}$?
- Is there a hybrid strategy (e.g., "partial" resolution + b-functions)?