The long exact sequence of Tor (a worked example)

Consider (*) $0 \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/8 \xrightarrow{/2} \mathbb{Z}/2 \to 0$ in $\mathbb{Z}$-mod (exact). Tensor with $\mathbb{Z}/2$:

$$\mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/2 \xrightarrow{/2} \mathbb{Z}/2 \to 0 \quad \text{, i.e.,}$$

(**) $$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \to 0 \quad \text{(no longer injective at the left)}$$

To complete this to a long exact sequence using Tor, choose projective resolutions for the "outside" groups in (*)

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & \mathbb{Z} \arrow[d, "\cdot 4"] & & \mathbb{Z} \arrow[d, "\cdot 2"] & \\
 & \mathbb{Z} \arrow[d, two heads, "/4"] & & \mathbb{Z} \arrow[d, two heads, "/2"] & \\
0 \arrow[r] & \mathbb{Z}/4 \arrow[r, "\cdot 2"] & \mathbb{Z}/8 \arrow[r, "/2"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Then, fill in the middle with their direct sum:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d] & 0 \oplus 0 \arrow[r, two heads, "\mathrm{proj}_2"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d, "\cdot 4"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, two heads, "\mathrm{proj}_2"] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d, two heads, "/4"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, two heads, "\mathrm{proj}_2"] & \mathbb{Z} \arrow[r] \arrow[d, two heads, "/2"] & 0 \\
0 \arrow[r] & \mathbb{Z}/4 \arrow[r, "\cdot 2"] & \mathbb{Z}/8 \arrow[r, "/2"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Next we apply the Horseshoe Lemma to build a resolution in the middle. This means we use projectiveness to to define green and blue maps, starting at the bottom, as follows:

1) Choose the bottom green map so that the diagram commutes.
2) Determine the bottom blue map so that the diagram commutes.
3) Choose the next green map into the kernel of the previous blue map.
4) Determine the next blue map so that the diagram commutes.
5) Repeat 3-4.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d] & 0 \oplus 0 \arrow[r, two heads, "\mathrm{proj}_2"] \arrow[d, "(0{,}0) \mapsto (0{,}0)"] & 0 \arrow[r] \arrow[d] \arrow[dl, "0"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d, "\cdot 4"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, two heads, "\mathrm{proj}_2"] \arrow[d, "(c{,}d) \mapsto (4c+d{,}\,2d)"] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot 2"] \arrow[dl, "\cdot(1{,}2)"'] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d, two heads, "/4"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, two heads, "\mathrm{proj}_2"] \arrow[d, two heads, "(a{,}b) \mapsto 2a+b"] & \mathbb{Z} \arrow[r] \arrow[d, two heads, "/2"] \arrow[dl, "18"'] & 0 \\
0 \arrow[r] & \mathbb{Z}/4 \arrow[r, "\cdot 2"] & \mathbb{Z}/8 \arrow[r, "/2"] & \mathbb{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

Now, forget the green maps and tensor with $\mathbb{Z}/2$:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d] & 0 \oplus 0 \arrow[r, two heads, "\mathrm{proj}_2"] \arrow[d, "(0{,}0) \mapsto (0{,}0)"] & 0 \arrow[r] \arrow[d] & 0 & \text{(row 2)} \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d, "\cdot 2"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, two heads, "\mathrm{proj}_2"] \arrow[d, "(c{,}d) \mapsto (d{,}0)"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\cdot 2"] & 0 & \text{(row 1)} \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, hook, "\mathrm{incl}_1"] \arrow[d, two heads, "/2"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, two heads, "\mathrm{proj}_2"] \arrow[d, two heads, "(a{,}b) \mapsto b"] & \mathbb{Z}/2 \arrow[r] \arrow[d, two heads, "/2"] & 0 & \text{(row 0)} \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "1"] & \mathbb{Z}/2 \arrow[r] & 0 & \text{(0th homologies)}
\end{tikzcd}$$

The homologies of the vertical complexes (with the black bottom row omitted) are $\mathrm{Tor}_n(\mathbb{Z}/4, \mathbb{Z}/2)$, $\mathrm{Tor}_n(\mathbb{Z}/8, \mathbb{Z}/2)$ and $\mathrm{Tor}_n(\mathbb{Z}/2, \mathbb{Z}/2)$. These we could have computed separately from any resolutions for $\mathbb{Z}/4$, $\mathbb{Z}/8$ and $\mathbb{Z}/2$ we wanted (no Horseshoe setup).

The snake lemma maps give us a long exact sequence of homologies (Tors).
(this is the part we needed the Horseshoe setup for):

$$\cdots 0 \xrightarrow{\delta} \mathrm{Tor}_1(\mathbb{Z}/4, \mathbb{Z}/2) \to \mathrm{Tor}_1(\mathbb{Z}/8, \mathbb{Z}/2) \to \mathrm{Tor}_1(\mathbb{Z}/2, \mathbb{Z}/2) \xrightarrow{\delta} \mathbb{Z}/4 \otimes \mathbb{Z}/2 \to \mathbb{Z}/8 \otimes \mathbb{Z}/2 \to \mathbb{Z}/2 \otimes \mathbb{Z}/2 \to 0$$

$$\cdots 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\mathrm{incl}_1} \mathbb{Z}/2 \oplus 0 \xrightarrow{\mathrm{proj}_2} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$

$$\cdots 0 \longrightarrow \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \longrightarrow 0$$