

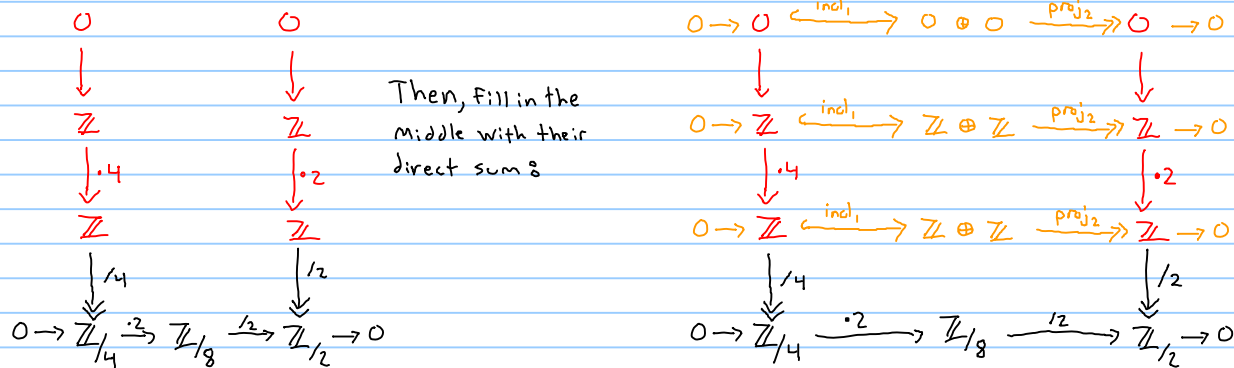
# The long exact sequence of Tor (a worked example)

Consider (\*)  $0 \rightarrow \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/8 \xrightarrow{\cdot 2} \mathbb{Z}/2 \rightarrow 0$  in  $\mathbb{Z}$ -mod (exact). Tensor with  $\mathbb{Z}/2$ :

$$\mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/2 \xrightarrow{\cdot 2} \mathbb{Z}/2 \rightarrow 0, \text{ i.e.,}$$

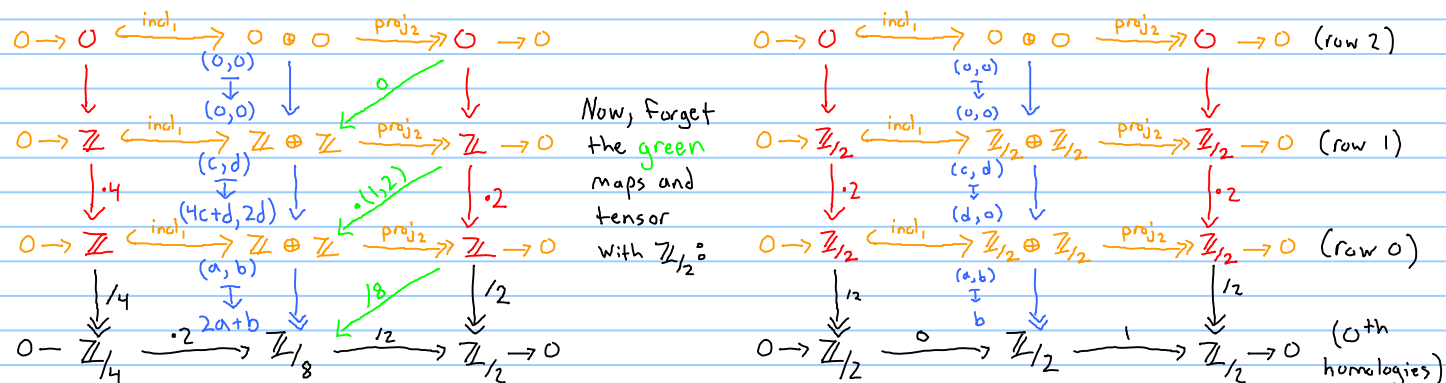
$$(**) \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \rightarrow 0 \text{ (no longer injective at the left)}$$

To complete this to a long exact sequence using Tor, choose projective resolutions for the "outside groups in (\*)



Next we apply the Horseshoe Lemma to build a resolution in the middle. This means we use projectiveness to define green and blue maps, starting at the bottom, as follows:

- 1) Choose the bottom green map so that the diagram commutes.
- 2) Determine the bottom blue map so that the diagram commutes.
- 3) Choose the next green map into the kernel of the previous blue map.
- 4) Determine the next blue map so that the diagram commutes.
- 5) Repeat 3-4.



The homologies of the vertical complexes (with the black bottom row omitted) are  $Tor_n(\mathbb{Z}/4, \mathbb{Z}/2)$ ,  $Tor_n(\mathbb{Z}/8, \mathbb{Z}/2)$  and  $Tor_n(\mathbb{Z}/2, \mathbb{Z}/2)$ . These we could have computed separately from any resolutions for  $\mathbb{Z}/4$ ,  $\mathbb{Z}/8$  and  $\mathbb{Z}/2$  we wanted (no Horseshoe setup).

The snake lemma maps give us a long exact sequence of homologies (Tors). (this is the part we needed the Horseshoe setup for):

$$\begin{aligned} \dots 0 &\xrightarrow{\delta} Tor_1(\mathbb{Z}/4, \mathbb{Z}/2) \rightarrow Tor_1(\mathbb{Z}/8, \mathbb{Z}/2) \rightarrow Tor_1(\mathbb{Z}/2, \mathbb{Z}/2) \xrightarrow{\delta} \mathbb{Z}/4 \otimes \mathbb{Z}/2 \rightarrow \mathbb{Z}/8 \otimes \mathbb{Z}/2 \rightarrow \mathbb{Z}/2 \otimes \mathbb{Z}/2 \rightarrow 0 \\ \dots 0 &\xrightarrow{\text{incl}_1} \mathbb{Z}/2 \xrightarrow{\text{incl}_1} \mathbb{Z}/2 \oplus 0 \xrightarrow{\text{proj}_2} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \rightarrow 0 \\ \dots 0 &\xrightarrow{1} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{1} \mathbb{Z}/2 \rightarrow 0 \end{aligned}$$