Stalk-local detection of irreducibility

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I think this is one more "stalk-local detection" result that people should learn right away:

Theorem. *A locally Noetherian scheme X is irreducible if and only if it is connected and stalk-locally irreducible. In particular, a Noetherian ring A is a domain if and only if it has no idempotents (i.e. it is not a direct product) and $A_{\mathfrak{p}}$ is a domain for every prime $\mathfrak{p}$.*

Motivation. A topological space is irreducible if and only if it is connected and locally irreducible. Thus, irreducibility of a connected space can be detected locally. For example, reducibility of the variety $xy = 0$ in the affine plane can be detected in any neighborhood of the origin. It turns out that in the case of a connected locally Noetherian scheme, it can even be detected stalk-locally.

The following lemma will reduce the problem to the affine case:

Lemma. *Any locally Noetherian scheme X is* affine-base-locally connected*, i.e. every point has a neighborhood base consisting of affine connected opens.*

Proof. Say $x \in X$ is contained in an open affine $\operatorname{Spec} A$, so $A = A_0$ is a Noetherian ring. If $\operatorname{Spec} A_{i-1}$ is not connected, write $A_{i-1} = A_i \times B_i$ (a direct product) where $\operatorname{Spec} A_i$ contains x. By Noetherianity, the ascending chain of ideals $C_i = B_1 \times \ldots \times B_i$ must stabilize, meaning eventually $\operatorname{Spec} A_i$ is a connected affine open neighborhood of x (contained in $\operatorname{Spec} A$ if this is of interest).

The affine case proven below will now be enough to prove the main result as follows: about each point of X, choose a connected neighborhood $\operatorname{Spec} A$

with A Noetherian, and stalk-local irreducibility will imply that $\operatorname{Spec} A$ is irreducible. Hence X is connected and locally irreducible, therefore irreducible by general topology.

So! To the affine case:

Proposition. *Suppose that A is a Noetherian ring. If $\operatorname{Spec} A$ is connected and stalk-locally irreducible, then it is irreducible.*

Proof. Let N be the nilradical of A. Suppose $a, b \in A$ and ab is nilpotent, which is to say that $ab \equiv 0 \pmod{N}$.

Stalk-local irreducibility of $\operatorname{Spec} A$ means precisely that for every prime p of A, the nilradical of A_p is prime. Thus, in each A_p either a or b is nilpotent. That is, for each p there is some $s \in A \setminus p$ and some n such that $sa^n = 0$ or $sb^n = 0$ in A, so that either sa or sb is nilpotent. Therefore the ideal $(N : a) + (N : b)$ has elements outside every prime (the s's), so it is the unit ideal and we can write $x + y = 1$ with $xa \equiv 0$ and $yb \equiv 0 \pmod{N}$. We can replace a by $a_1 = y$, and note that $a = (x + a_1)a \equiv a_1 a$ (mod N), so the ideal $(a_1) + N$ contains the ideal $(a) + N$. We can continue inductively to get an ascending chain of ideals

$$(a_1) + N \subseteq (a_2) + N \subseteq (a_3) + N \subseteq \dots$$

with the property that $a_i b \equiv 0$ and $a_{i+1} a_i \equiv a_i$. This chain will stabilize, so eventually we get $a_{n+1} \equiv a_n r$ for some $r \in A$, whence $a_{n+1}^2 \equiv a_{n+1} a_n r \equiv a_n r \equiv a_{n+1}$. But connectedness of $\operatorname{Spec} A$ is precisely the condition that A/N has no idempotents except 1 and 0, so either $a_{n+1} \equiv 1$ in which case $b \equiv 0$, or $a_{n+1} \equiv 0$ in which case $a \equiv 0$. Thus, N must be prime, i.e. $\operatorname{Spec} A$ is irreducible. □

As an application[1] of the above, we can remove the "integral" hypothesis from Hartshorne's *Algebraic Geometry* Theorem II.6.11:

Theorem. *If a scheme X is Noetherian, separated, and locally factorial, then $\operatorname{Weil}(X) \simeq \operatorname{Cart}(X)$, i.e. its Weil divisor group is isomorphic to its Cartier divisor group.*

Proof. Since X is Noetherian, by the lemma we know it is locally connected, so we can break it into finitely many (open) connected components

[1] added March 23, 2009

$X_1, \ldots, X_n$. The stalks of X_i are UFDs, hence irreducible, so X_i is irreducible by stalk-local detection, and reduced by the same argument. Hence, each X_i is a domain (i.e. an "integral" scheme) where we can apply AG II.6.11. Then we have $\mathrm{Weil}(X) \simeq \prod_i \mathrm{Weil}(X_i) \simeq \prod_i \mathrm{Cart}(X_i) \simeq \mathrm{Cart}(X)$, as required. $\square$