ELECTRIC NETWORK SYNTHESIS

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This document summarizes our understanding of the problems presented by Rudolf Kalman in his lecture in the Berkeley Algebraic Statistics Seminar on October 26, 2011. We consider electrical networks which can be defined mathematically as follows. An *RLC network* is an undirected connected graph $\Gamma = (V, E)$ satisfying the following properties:

(1) There are no self loops but parallel edges are allowed.
(2) Two of the nodes are distinguished.
(3) Every edge is labeled by an unknown real parameter times either s^{-1}, s^0 or s^1 where s is an indeterminate.

In electrical network theory [2,3], the edges of Γ correspond to resistors (R), inductors (L) and capacitors (C) that attain certain positive real values, and we want to study the impedance (defined below) between the two distinguished nodes. Two basic questions are:

- Given Γ, what is the impedance between the two distinguished nodes? (Prediction)
- Given an impedance function, does there exist a network with that impedance, and if so what is the minimal such network? (Synthesis)

If Z is the impedance of a component (or network), then $Y = \frac{1}{Z}$ is called the *admittance*. Consider the simplest network with precisely two components of the same type. There are only two graphs – either a loop (two components in parallel) or a chain (two components in series). There are only two rules for how to compute the impedance Z of the network, depending on the type of component. For resistors and inductors impedance adds in series; $Z = Z_1 + Z_2$, and admittance adds in parallel; $\frac{1}{Z} = \frac{1}{Z}_1 + \frac{1}{Z}_2$. For capacitors the reverse is true, namely the impedance of capacitors adds in parallel and the admittance adds in series.

A network is *series-parallel* if it is inductively constructed from sub-networks that are all either in series or parallel. Such networks are nice because their impedance can be computed inductively. However, most networks (including our Example 2 below) are not series-parallel. In that general case, we define the impedance between the two distinguished nodes is the sum of the impedance of each path in the graph from one distinguished node to the other. As we shall see, this infinite sum can be written conveniently as a rational generating function.

Consider any graph $\Gamma = (V, E)$ satisfying conditions (1)-(3) above. Set $V = \{1, 2, \ldots, d\}$, assume the distinguished nodes are 1 and 2, and represent Γ by its Laplacian matrix Λ_Γ. This is the symmetric $d \times d$-matrix of rank $d-1$ whose columns sum to zero and whose off-diagonal entries are the labels of the edges. By the Matrix Tree Theorem, all $(d-1)\times(d-1)$-subdeterminants of Λ_Γ are equal up to sign. Explicitly, that common comaximal minor $f_\Gamma(s)$ is the sum over all spanning trees T of the products of the edge labels in T. Thus $f_\Gamma(s)$ is a Laurent polynomial in s whose coefficients are sums of monomials in the $|E|$ positive real parameters. Let $\Gamma[1 = 2]$ denote the graph obtained from Γ by identifying the two distinguished nodes. Thus $\Gamma[1 = 2]$ is the graph on $d - 1$ nodes whose Laplacian $\Lambda_{\Gamma[1=2]}$ is obtained from Λ_Γ by adding the first and second rows. We write $f_{\Gamma[1=2]}(s)$ for the Laurent polynomial that is obtained by taking any comaximal subdeterminant of Λ_Γ The *impedance* of the network Γ is the rational function

$$z_\Gamma(s) \quad = \quad \frac{a_\Gamma(s)}{b_\Gamma(s)} \quad = \quad \frac{f_{\Gamma[1=2]}(s)}{f_\Gamma(s)}.$$

Here $a_\Gamma(s)$ and $b_\Gamma(s)$ are relatively prime polynomials in s. The pair of nonnegative integers

$$(\alpha_\Gamma, \beta_\Gamma) \;=\; \big(\mathrm{degree}_s(a_\Gamma), \mathrm{degree}_s(b_\Gamma)\big)$$

is called the *bidegree* of the network Γ.

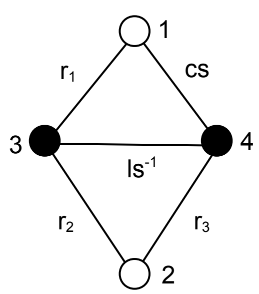
The set of all rational functions with that given numerator degree and denominator degree is the projective space $\mathbb{P}^{\alpha_\Gamma+\beta_\Gamma+1}$. The space of parameters for the network Γ is $\mathbb{R}^E_{>0}$. What we are interested in is the polynomial map that takes the parameters to the impedance:

$$Z_\Gamma \,:\, \mathbb{R}^E \;\to\; \mathbb{P}^{\alpha_\Gamma+\beta_\Gamma+1}, \text{ edge weights } \mapsto \frac{a_\Gamma(s)}{b_\Gamma(s)}$$

The network Γ is *minimal* if the map Z_Γ is finite-to-one. A necessary condition for minimality is that $|E| = \alpha_\Gamma + \beta_\Gamma + 1$. Here is the problem we wish to solve for any fixed bidegree (α, β):

Problem 1. (1) *Compile the finite list of all graphs Γ satisfying the necessary condition*
(2) *Among all networks Γ which do not have series-parallel reductions to simpler ones, identify those that are minimal and compute their impedances $z_\Gamma(s)$.*
(3) *For each minimal network Γ, characterize the set $Z_\Gamma(\mathbb{R}^E_{>0})$, i.e. the image of the orthant of positive parameter values, as a semi-algebraic subset of $\mathbb{P}^{\alpha_\Gamma+\beta_\Gamma+1}$.*
(4) *By intersecting the various $Z_\Gamma(\mathbb{R}^E_{>0})$, determine the stratification of the space $\mathbb{P}^{\alpha_\Gamma+\beta_\Gamma+1}_{>0}$ of positive rational functions according to sets of networks that realize them.*
(5) *For each minimal Γ, find the algebraic degree of the map ψ_Γ, and compute the inverse as an algebraic function. (This inverse is rational if the algebraic degree is 1).*
(6) *Conjecture: The inverse to Z_Γ is rational if and only if Γ is series-parallel.*

Example 2. Consider the following RLC network Γ with real parameters r_1, r_2, r_3, l, c. The white nodes represent the distinguished nodes 1 and 2. The impedance of Γ is given by



$$z_\Gamma(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}$$

$$\begin{aligned} a_2 &= c(r_1 + r_2) \\ a_1 &= cl + r_3(r_1 + r_2) \\ a_0 &= l(r_1 + r_2 + r_3) \\ b_2 &= c(r_1 r_2 + r_2 r_3 + r_1 r_3) \\ b_1 &= cl(r_2 + r_3) + r_1 r_2 r_3 \\ b_0 &= l r_1 (r_2 + r_3) \end{aligned}$$

These formulas specify the map $Z_\Gamma : \mathbb{R}^5 \to \mathbb{P}^5$, $(r_1, r_2, r_3, l, c) \mapsto (a_2 : a_1 : a_0 : b_2 : b_1 : b_0)$. The graph of this map is the variety defined by the 2×2-minors of the matrix

$$\begin{bmatrix} a_2 & a_1 & a_0 & b_2 & b_1 & b_0 \\ c(r_1{+}r_2) & cl{+}r_3(r_1{+}r_2) & l(r_1{+}r_2{+}r_3) & c(r_1r_2{+}r_2r_3{+}r_1r_3) & cl(r_2{+}r_3){+}r_1r_2r_3 & lr_1(r_2{+}r_3) \end{bmatrix}$$

For fixed generic values of $a_2, a_1, a_0, b_2, b_1, b_0$, these equations have precisely four complex solutions, so the algebraic degree of the map Z_Γ is 4. This means that the network Γ is minimal, and we can express the inverse to Z_Γ in radicals using Cardano's formula. $\square$

In the applications of [1–3], the edges labeled by some cs, ls^{-1} and r correspond to electrical components with capacitance $C = c$, inductance $L = l^{-1}$ and resistance $R = r^{-1}$ respectively. Here we care about the inverse to Z_Γ only on the semi-algebraic set corresponding to positive parameter values, and our Problem 1 (2) is aimed at characterizing that inversion domain as a semi-algebraic set. Rudolf Kalman pointed out that the resultant and its Sylvester matrix of the two polynomials $a_\Gamma(s)$ and $b_\Gamma(s)$ should play a special role in that characterization. He mentioned that the physically interesting cases are when $|\alpha - \beta| \leq 1$, the total number of capacitors and inductors equals $\max(\alpha, \beta)$, and the coefficients of $a_\Gamma(s)$ and $b_\Gamma(s)$ are positive. Following are two Theorems* from Professor Kalman's talk, which we should regard as true for bidegree $(2, 2)$, with proof by exhaustion, and as conjectures for larger bidegrees.

Theorem 3. *The following are equivalent for a RLC network Σ with associated graph Γ_Σ:*

(1) Γ_Σ *is simple series parallel*
(2) *The resultant* $\mathrm{Res}_s(a, b)$ *is a product of monomials in the parameters.*
(3) *Each coordinate of the inverse to Z is expressible as a ratio of "invariants" (entries of the adjoint of the Sylvester matrix).*

In Example 2, the Sylvester resultant of the two quadrics equals, in parameters,

$$\mathrm{Res}(a, b) \;=\; lc(lcr_2r_3 + r_1r_2r_3^2 + r_2^2lc + r_1^2r_3^2)^2.$$

This is not a product of monomials in r, l, c because Γ is not series parallel. Note that *simple series parallel* is the condition that the maximum length of an elementary cycle Γ_Σ is 2. We say that $z(s)$ is *realizable* if $z(s) = z_\Gamma(s)$ for some simple series-parallel network Γ.

Theorem 4. *Let $z(s)$ be a rational function of bidegree $(2, 2)$ with positive coefficients. Then*

$$Res(a, b) > 0 \text{ and } a_0b_2 - a_2b_0 > 0 \text{ iff } Z \text{ is RC-realizable},$$
$$Res(a, b) > 0 \text{ and } a_0b_2 - a_2b_0 < 0 \text{ iff } Z \text{ is RL-realizable}.$$

The general realizability question has been solved also.

Theorem 5. *A given impedance function $z(s)$ is RC-realizable or RL-realizable by a simple series-parallel network if and only if all roots of $a(s)$ and $b(s)$ are real and they interlace.*

Note that Lemma 5 [4] tells us that the interlacing condition on $a(s)$ and $b(s)$ are satisfied if and only if the Bezoutian matrix is positive definite.

Conclusion: We are looking for a grad student who will solve Problem 1 for $(\alpha, \beta) = (3, 3)$.

References

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